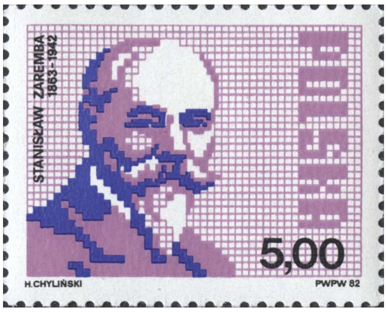
STEFAN BANACH (March 30, 1892 – August 31, 1945)

by HEINZ KLAUS STRICK, Germany

STEFAN BANACH was born in Kraków (then part of Austria–Hungary); an illegitimate child, his mother disappeared without a trace the day after the four-day-old infant was baptized. The father, after whom he was named, gave the child to its grandmother to be reared, and when she became ill, the child ended up in a foster family, which took good care of the child, among other things giving him an early opportunity to learn French. After attending primary school, he went on to secondary school, where he met WITOLD WILKOSZ, who, like BANACH, would someday become a professor of mathematics. Indeed, mathematics was the only thing in which the two young men were interested, but when they completed high school in 1910, they both decided against further studies in mathematics, believing that there was nothing new to be discovered in the field. BANACH began a course in engineering, while his friend took up studies in Oriental languages.

Since his father had abdicated all financial support for his son, STEFAN BANACH no longer had a reason to remain in Kraków. He moved to Lwów (today the Ukrainian Lwiw) to begin his studies at the technical university. He made slow progress, since to support himself, he worked many hours as a tutor. In 1914, after an interim exam, he broke off his studies following the outbreak of World War I. When Russian troops invaded Lwów, he returned to Kraków. Because of his poor eyesight, he was rejected for service in the armed forces, but was sent to work building roads; he also taught mathematics and eventually was able to attend lectures on mathematics, probably given by STANISŁAW ZAREMBA.



A chance meeting in a park brought him acquaintance with HUGO DYONIZY STEINHAUS, who following the receipt of his doctorate under DAVID HILBERT was working on his habilitation. He told STEFAN BANACH about a mathematical problem on which he was deeply engaged.

Only a few days later, STEFAN BANACH had found a solution to the problem, and together they wrote a paper for a professional journal, whose publisher was ZAREMBA. Because of the war, the paper was not published until 1918; in the meantime, however, BANACH wrote one mathematical article after another. After the end of the war and the creation of a newly independent Polish state, STEINHAUS and ZAREMBA founded the *Kraków Mathematical Society* (after 1920, the *Polish Mathematical Society*), whose chairmanship was taken by ZAREMBA. BANACH obtained a teaching position at the technical university in Lwów.

Although he had no university degree, after submitting an article with the title "*Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*" ("*On operations in abstract sets and their application to integral equations*"), he was awarded a doctorate. This work is considered to represent the origin of a new field of mathematics: *functional analysis*.

Beginning with vector spaces whose elements are functions, BANACH investigated the properties of spaces in which a norm $\|\ \|$ can be defined: by defining $d(a,b)=\|a-b\|$, it is possible to measure the distance between elements *a*, *b*.

BANACH's ground-breaking work dealt with the properties of complete metric spaces X, that is, spaces in which CAUCHY sequences in X have their limit in X. A short while thereafter, such spaces began to be called BANACH spaces, and the vocabulary and terminology used by BANACH in his work were taken up by other mathematicians.

In quick succession there followed new articles by BANACH on the abstract spaces that he was investigating. By 1922, he had completed his habilitation and was named an associate professor. After an academic year in Paris, he continued his work as a professor in Lwów, founding a new mathematical journal with STEINHAUS, which was devoted in particular to the further development of functional analysis, and wrote a number of successful mathematical textbooks. In 1932 he published what became the standard text in functional analysis, the "*Théorie des opérations linéaires*".

BANACH's working method was unusual in that while other mathematicians prefer the quietness of a study or library, BANACH preferred the bustling atmosphere of a café, where he could concentrate on his thoughts, unperturbed by the noise and activity around him. When his favourite café, the Scottish Café (*Kawiarnia Szkocka*) closed, he would often go to the cafeteria at the railway station, which was open at all hours.

In 1939, BANACH was elected president of the *Polish Mathematical Society*. After the start of World War II, Soviet troops occupied Lwów following the signing of the MOLOTOW-RIBBENTROP pact. Since BANACH had spent considerable time in Moscow during the 1930s and had good connections with Soviet mathematicians, he was able to continue his work, and was even named a dean of the university. When the German troops occupied the city in 1941, the situation became difficult for BANACH. It was only by chance that he escaped the mass murder of Polish intellectuals by the SS. After the liberation of Lwów in July 1944, the renewal of academic work in Polish universities was supported by Soviet mathematicians. BANACH, however, had only a short time remaining for his work: weakened by the privations of the German occupation, he had no resistance left, and in August 1945, STEFAN BANACH died of lung cancer.

Among the most important theorems formulated by BANACH is the theorem that today goes under the name *BANACH fixed-point theorem*: If for a mapping *f* of a metric space IR into itself there exists a $q \in [0,1[$ such that for all a, b, one has $d(f(a),f(b)) \leq q \cdot d(a,b)$, then *f* has a fixed point; that is, there exists a number $x_{fix} \in IR$ such that $f(x_{fix}) = x_{fix}$.

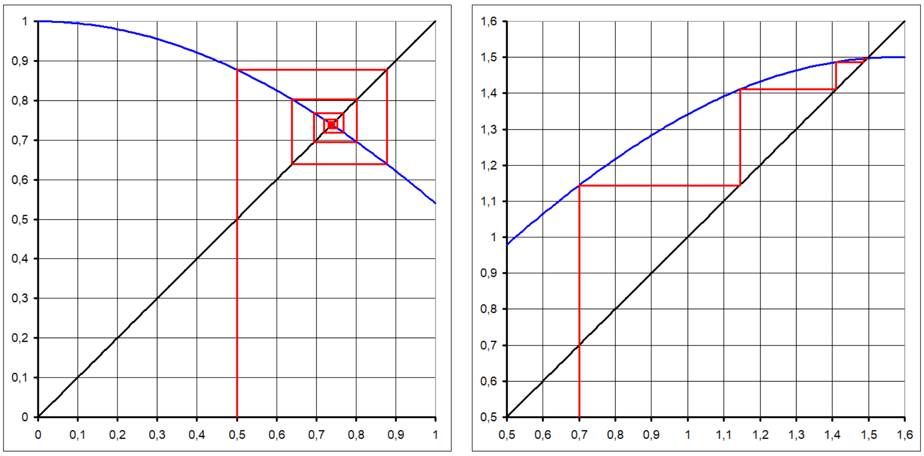
An important application is the following: If *f* is a differentiable real-valued function that maps the interval $[a,b]$ into itself and if there exists a number $q < 1$ such that for all $x \in [a,b]$ one has $|f'(x)| \leq q$, then the assumptions of the BANACH fixed-point theorem are satisfied. For by the *mean value theorem* of differential calculus, there exists a number $c \in]a,b[$ such that $f(b) - f(a) = f'(c) \cdot (b-a)$, and if $|f'(x)| \leq q < 1$ for all $x \in [a,b]$ this holds in particular for the number c. Hence $|f(b) - f(a)| \leq |b-a|$.

From the equation $x_{n+1} = f(x_n)$, a recurrent sequence is defined that converges to the fixed point, which exists by the *BANACH fixed-point theorem*. There is a concrete estimate for the distance between the *n*th term of the recurrently defined sequence and the limiting value x_{fix}, the fixed point: $|x_n - x_{fix}| \leq \frac{q^n}{1-q} \cdot |x_1 - x_0|$

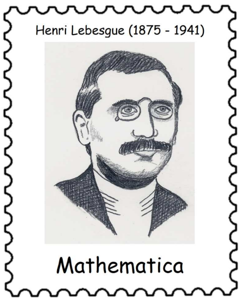
Examples:

The graph of the function *f* with $f(x) = \cos(x)$ has a fixed point in the interval $[0,1]$: namely, one has for the derivative that $f'(x) = -\sin(x)$: $|f'(x)| = |-\sin(1)| < 0.85$. Therefore, the sequence defined by $x_{n+1} = \cos(x_n)$ has a limit $x_{fix} = 0.7390... = \cos(0.7390...)$.

For the derivative of $f(x) = 0.5 + \sin(x)$, one has in the interval $[0.5, 1.5]$ that $|f'(x)| = |\cos(0.5)| < 0.88$. Therefore, the sequence defined by $x_{n+1} = 0.5 + \sin(x_n)$ has a limit, which is given by $x_{fix} = 1.4973... = 0.5 + \sin(1.4973...)$.



In collaboration with other mathematicians, BANACH discovered further mathematical theorems of enormous importance, such as the HAHN-BANACH theorem, the BANACH-STEINHAUS theorem, and many others.



(drawing: © Andreas Strick)

In 1924, together with the mathematician ALFRED TARSKI, who was from Warsaw, he discovered a paradox of set theory (the *Banach-Tarski paradox*) that contradicts an apparent fact about volumes ("*Sur la décomposition des ensembles de points en parties respectivement congruentes*"). According to the expanded definition of the integral put forward by the French mathematician HENRI LEBESGUE at the beginning of the twentieth century, it is possible to give a “measure” to rather complicated sets of points. BANACH and TARSKI determined that it is possible to decompose a sphere into point sets resembling dust clouds in such a way that they can be assembled into two spheres each with the same volume as the original sphere. That is, the volume of the sphere is doubled.

When STEINHAUS (1887–1972) was once asked what his greatest accomplishment was, he replied, “BANACH was my greatest scientific discovery.”

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Translated by David Kramer

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