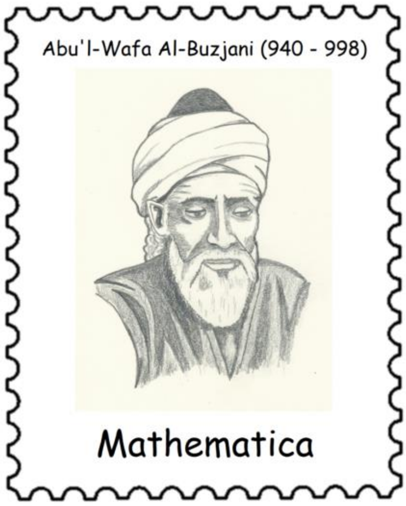
ABU'L-WAFA AL-BUZJANI (940 – 998)

by HEINZ KLAUS STRICK, Germany



(drawing © Andreas Strick)

The conquest of Baghdad in 945 marked the beginning of the Buyid dynasty's rule in what is now Iraq and the western part of Iran. The regency lasted just over 100 years.

The new caliph AHMAD BUYEH and his descendants saw themselves as patrons of the arts and sciences. They had a large observatory built in the palace garden, including a 6-metre-long quadrant and an 18-metre-long walled sextant, which made it possible to take more precise measurements than before.

One of the most important scientists working in Baghdad at this time was MOHAMMAD ABU'L-WAFA AL BUZJANI, who grew up in Buzjan (Khorasan province) and received a comprehensive education as the child of a wealthy family.

At the age of 19, he went to Baghdad and was first commissioned to translate the writings of EUCLID and DIOPHANTUS and he then wrote commentaries on AL-KHWARIZMI's works.

In the meantime, the use of *Indian numerals* had become widespread in scientific circles, but in everyday life the technique of *finger-calculation* without the use of a numerical notation was still predominantly used. From time immemorial, finger arithmetic had used the thumb as a pointer and the 4×3 joints of the other fingers of the same hand to indicate the numbers from 1 to 12; the fingers of the other hand were used to indicate the numbers from 13 to 60.

For merchants and scribes, ABU'L-WAFA AL-BUZJANI wrote a book on what arithmetic skills were necessary in everyday life. All the numbers appearing in the book were noted as words, and even complicated calculations were described verbally.

The book comprised seven chapters. The first chapter dealt with fractions. In everyday life, the common fractions $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}$ played a special role; other fractions were written down as multiples of these common fractions.

Since both the units of measurement and money were expressed in the *sexagesimal system* (i.e. with a base of 60) originating from the Babylonians, e.g. 1 *dirham* = 60 *fulus*, calculating with fractions was easy if the denominators were divisors of 60:

dirham	1/2	1/3	1/4	1/5	1/6	1/8	1/9	1/10	1/12	1/15	1/16	1/18	…
fulus	30	20	15	12	10	7;30	6;40	6	5	4	3;45	3;20	

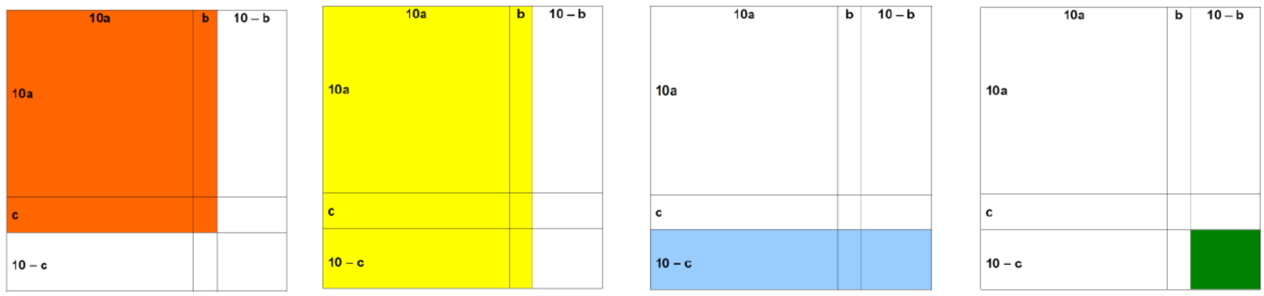
Approximations were used for fractions with other denominators; for example, ABU'L-WAFA gave the following approximations for the fraction $\frac{3}{17}$:

$$\frac{3}{17} \approx \frac{3+1}{17+1} = \frac{2}{9} \quad \text{as well as} \quad \frac{3}{17} \approx \frac{3+\frac{1}{2}}{17+\frac{1}{2}} = \frac{1}{5} \quad \text{and even} \quad \frac{3}{17} \approx \frac{3+\frac{1}{7}}{17+\frac{1}{7}} = \frac{11}{60} = \frac{1}{6} + \frac{1}{60}.$$

In the second chapter, arithmetic with whole numbers and fractions was explained. Among other things, ABU'L-WAFA gave a seemingly complicated arithmetic trick for multiplying whole numbers in the same decade (i.e. $1 \le b, c \le 9$):

$$(10a+b)\cdot(10a+c) =$$
$$10(a+1)\cdot[10a+b-\{10(a+1)-(10a+c)\}]+[10(a+1)-(10a+b)]\cdot[10(a+1)-(10a+c)]$$

The idea behind this becomes clear if you illustrate the factors and the corresponding areas with the help of rectangles: $(10a+b)\cdot(10a+c) =$ orange = (yellow – blue) + green



(10a+b)·(10a+c) (10a+b)·10·(a+1) 10·(a+1)·{10·(a+1) – (10a+c)} [10·(a+1) – (10a+b)] · [10·(a+1) – (10a+c)]

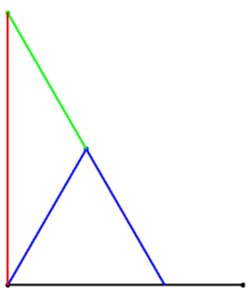
Example: *a* = 4, *b* = 3, *c* = 6: 43 · 46 = 50 · [43 – {50 – 46}] + [50 – 43] · [50 – 46] = 50 · 39 + 7 · 4

The trick even works for *a* = 0 even though here negative differences occur. These ABU'L-WAFA called *debts* without going into detail. It is the only place in the literature of medieval Islam where negative numbers are mentioned.

The other chapters dealt with the determination of surface areas and volumes as well as distances from inaccessible points, with the calculation of taxes, tolls, exchange rates and barter transactions.

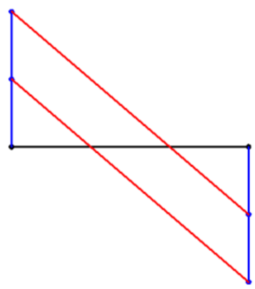
Another book, *What craftsmen need in terms of geometric constructions*, was written around 990. Here ABU'L-WAFA adopted numerous constructions by Greek mathematicians, but also developed many new ideas of his own.

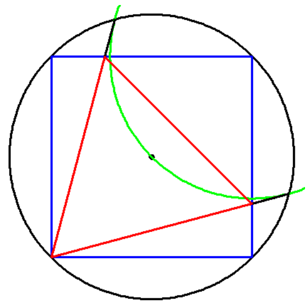
A special feature was his constructions with a ruler and a *rusted* compass, i.e. with a compass that cannot be adjusted. For craftsmen it is important that the radius does not change unintentionally during the work.



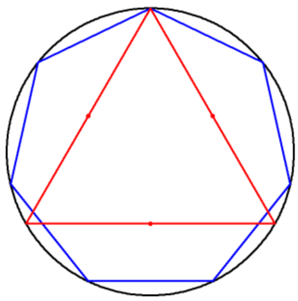
The diagram on the left shows how to create a perpendicular at the end of a line without extending the line itself,

the diagram on the right shows how to divide a line into three equal sections (using two perpendiculars).





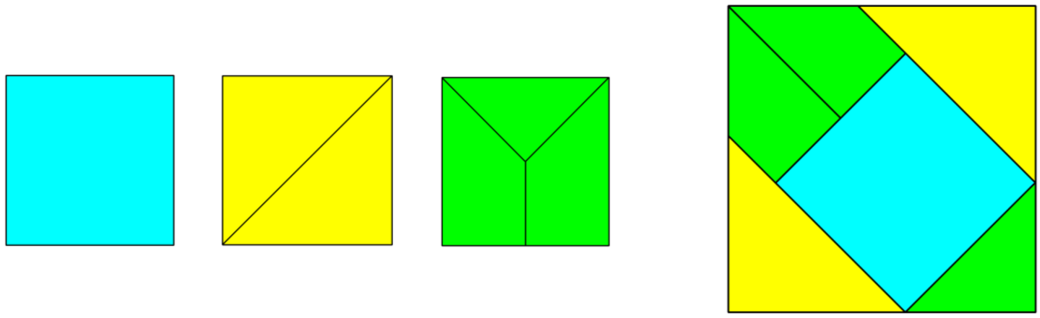
Abu'l-Wafa's construction of an equilateral triangle inscribed in a square is also original (diagram on the left).



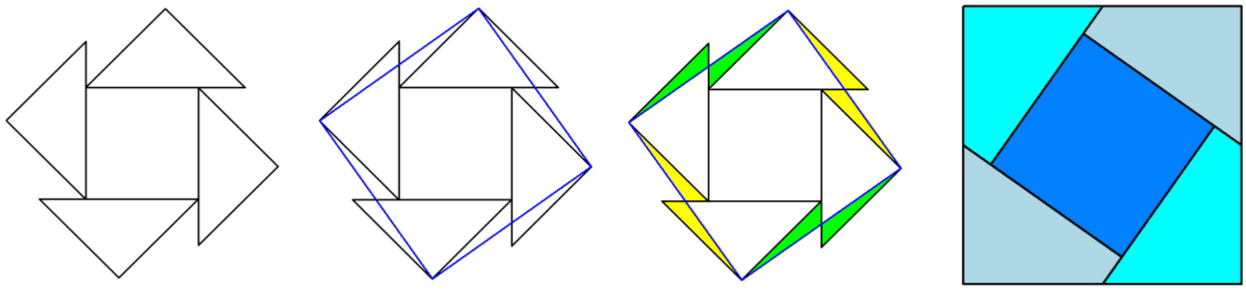
The diagram on the right shows the approximate determination of the side of a regular heptagon: First, one constructs an equilateral triangle into a given circle and then subtracts half the side length of the triangle as the side of the heptagon (this is $\frac{1}{2}\cdot\sqrt{3}\cdot r \approx 0.8660\cdot r$, while the correct length would be $0.8678\cdot r$).

He devoted part of the book to the geometric patterns used by craftsmen in the form of mosaics in decorating buildings. Since the creation of mosaics involved the skilful decomposition of polygons, he devoted a great deal of time to the problems that arose in the process.

For the task of dissecting three squares of equal size in such a way that their "puzzle pieces" together form a square again, the illustrated dissection was common among craftsmen. Although at first glance they seem to fit wonderfully, however, on closer calculation one can see the inaccuracy.



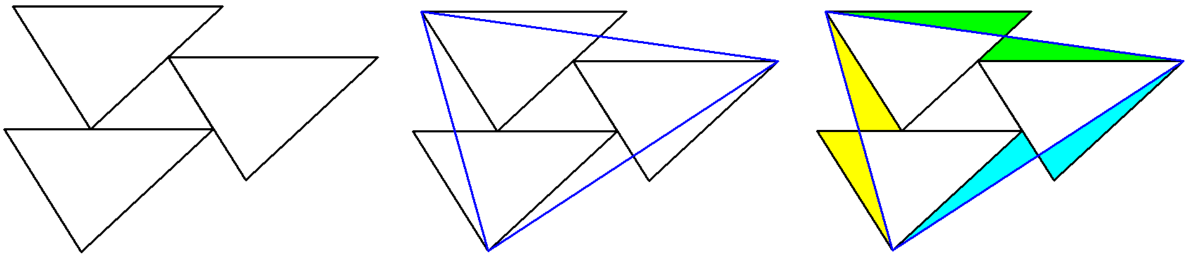
Abu'l-Wafa found an elegant solution: He halved two of the squares along a diagonal and put them together with the undivided square to form a figure. The blue connecting lines of the outer corner points of the figure then formed the square he was looking for. The protruding triangles fit exactly into the gaps inside the square; together with the white remaining triangles, they each form a special square with two right angles.



Such decompositions of a square can be found as tile patterns in numerous mosques, for example in the Friday Mosque of Isfahan (see picture on the right).

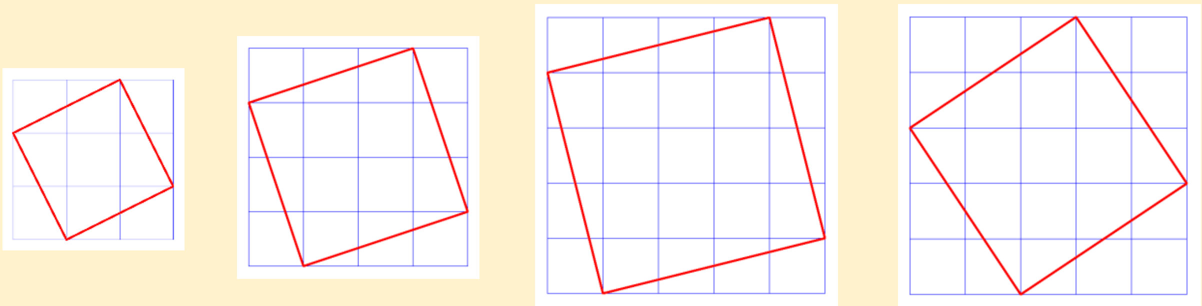
The decomposition is a special case of a general decomposition proof of the theorem of Pythagoras as stated by Henri Perigal in the 19th century.

Abu'l-Wafa found a similar solution for a pattern of three congruent triangles enclosing a smaller similar triangle. If one connects the outer corner points with each other, then a triangle is created that is equal in area to the four initial triangles.



For integers *a, b*, Abu'l-Wafa discovered a method of dividing a square with area $a^2 + b^2$ into smaller squares:

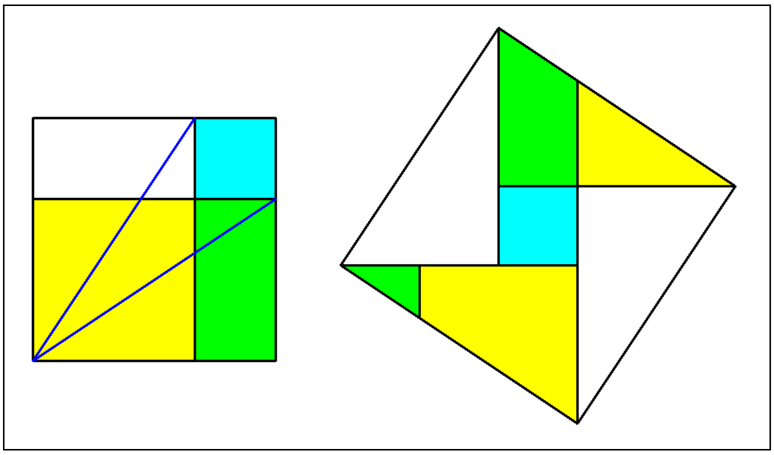
In the square grid laid over it, $a + b - \gcd(a, b)$ of the unit squares are divided; the parts each complement each other with a symmetrically lying puzzle piece again to form a unit square. Altogether the figure consists of $a^2 + b^2 + 2 \cdot (a + b - \gcd(a, b))$ puzzle pieces.



$a = 1$; $b = 2$ $a = 1$; $b = 3$ $a = 1$; $b = 4$ $a = 2$; $b = 3$

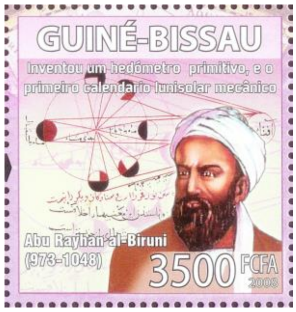
Abu'l-Wafa also gave a variant of the proof of Pythagoras's theorem:

Given two squares with side lengths *a* and *b*, where *a* < *b*. In the figure on the left, the square with side length *a* (yellow) is placed in one of the corners of the square with side length *b*; thus a square with side length *b* – *a* (turquoise) is created in the opposite corner. Then the puzzle pieces of a^2 and b^2 result in a square with side length *c* (= length of the blue lines).



A large part of the reputation that Abu'l-Wafa enjoyed for centuries in the Islamic world was also due to his extensive astronomical research, including his editing of Ptolemys's *Almagest*.

He was in contact with Abu Arrayhan al-Biruni and was able to determine the difference in the geographical longitudes of the two observation sites from the data he determined during a lunar eclipse in Kath (today Uzbekistan) and his own measurements.



ABU'L-WAFA is also the first to recognise the simplification when considering the sine, cosine and tangent of an angle on the unit circle (i.e. with radius $R = 1$ instead of $R = 60$, which had been usual until then). Instead of the chord tables that had been usual until then, he created tables with the values of all six trigonometric functions with a step size of 15 minutes and an accuracy of the equivalent of 8 decimal places.

He succeeded in doing this with the help of the addition theorem for sines and the half-angle theorem:

$\sin(\alpha \pm \beta) = \sin(\alpha) \cdot \cos(\beta) \pm \cos(\alpha) \cdot \sin(\beta)$ and $\cos(\alpha) = 1 - 2 \cdot \sin^2(\frac{\alpha}{2})$,

which he had derived in order to arrive at the values of smaller angles step by step from known values, such as sin(60°) and sin(72°).

He also discovered the sine theorem for spherical triangles,

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)}$$

where a, b, c are the side lengths and A, B, C the sizes of the opposite angles.

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