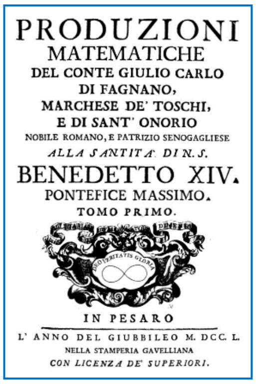
**GIULIO CARLO FAGNANO** (September 26, 1682 – May 18, 1766)

by HEINZ KLAUS STRICK, Germany


According to CARL GUSTAV JACOB JACOBI, December 23, 1751 was the birthday of the *elliptic functions*.

And so it was as follows: LEONHARD EULER had been asked to appraise the collected writings *Produzioni matematiche* by Count GIULIO CARLO FAGNANO, since he had been proposed as a member of the *Berlin Academy of Sciences*.

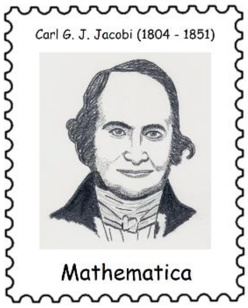

PRODUZIONI
MATEMATICHE
DEL CONTE GIULIO CARLO
DI FAGNANO,
MARCHESE DE' TOSCHI,
E DI SANT' ONORIO
NOBILE ROMANO, E PATRIZIO SENOGAGLIESE
ALLA SANTITA' DI N. S.
BENEDETTO XIV.
PONTEFICE MASSIMO.
TOMO PRIMO.
IN PESARO
L' ANNO DEL GIUBBILEO M. DCC. L.
NELLA STAMPERIA GAVELLIANA
CON LICENZA DE' SUPERIORI.

Among FAGNANO's contributions, EULER discovered the 1718 treatise *Metodo per misurare la Lemniscata* (Method for Measuring the Arc Length of a Lemniscate), which inspired him.

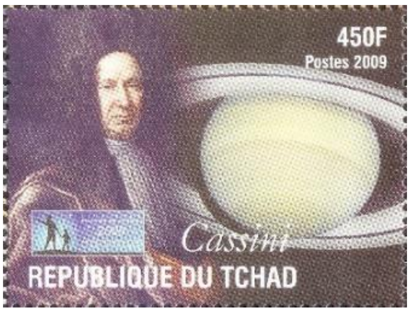
In fact, the Italian mathematician FAGNANO had managed to solve a problem that others had previously struggled to solve. EULER immediately recognized how FAGNANO's approach can be generalized; this EULER treatise then contributed significantly to the development of the theory of elliptic functions.

FAGNANO himself was so enthusiastic about the properties of the lemniscate he had discovered that he had the curve printed on the cover of the *Produzioni matematiche* and also decreed that it should be carved on his tombstone.

That curve was discovered by GIOVANNI DOMENICO CASSINI, who, around 1680, was generally concerned with the question of which curves result when the *product* of the distances from a curve point $P(x, y)$ to two points $P_1(+c, 0)$ and $P_2(-c, 0)$ is constant and equal to $a^2$ $(a, c \geq 0)$, i.e. the following applies: $(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4$.

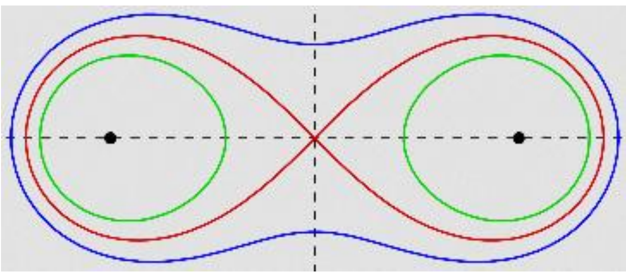


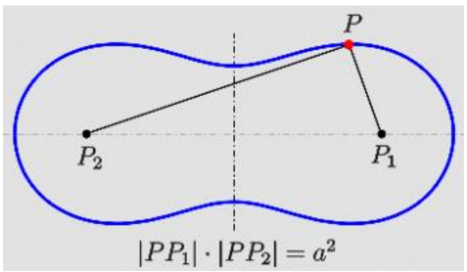






In the case $a = c$, you get the graph of the lemniscate drawn in red (Fig. Wikipedia).





**Arc length of curves – elliptic integrals**

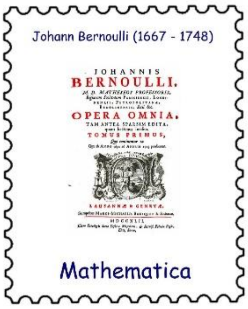
If a curve is given by the graph of a differentiable function *f*, then – after applying PYTHAGORAS's theorem – the length *L* of the arc between two points $(a, f(a))$ and $(b, f(b))$ can be determined using an integral: $L(a,b) = \int_a^b \sqrt{1+(f'(x))^2}\,dx$ .

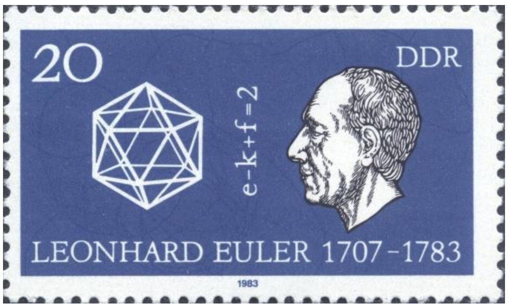
For example, a circle with $f(x) = \sqrt{r^2 - x^2}$ and $f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}$ results in the length of a quadrant: $L(0,r) = \int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = r \cdot \int_0^r \frac{dx}{\sqrt{r^2-x^2}} = r \cdot [\arcsin(1) - \arcsin(0)] = \frac{\pi}{2} \cdot r$ .

This calculation is possible because the integral can be described (one says: "given in closed form") by a concrete elementary function. The same applies to the cycloid, the logarithmic spiral and the parabola, but not to the calculation of the arc length of an ellipse or a hyperbola, that is, although you can express the arc length by an integral, you cannot perform a concrete calculation.

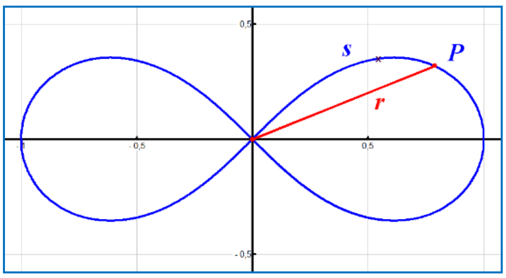
Generally, integrals are called *elliptic integrals* where the integrand function is a rational function containing a square root of a cubic or biquadratic term.

JACOB BERNOULLI, from whom the term *lemniscate* (Greek: loop) comes, had shown that determining the length of an arc from the origin to a point on the lemniscate leads to an integral of type $\int \frac{dx}{\sqrt{1-x^4}}$, which, however, cannot be given in closed form. Although he was able to represent this integral function by a series expansion, neither he nor his brother JOHANN nor EULER made any significant progress.


FAGNANO examined the arc length of the lemniscate with $(-1,0), (1,0)$ (i.e. $a = \frac{\sqrt{2}}{2}$) in the 1st quadrant, i.e. the upper arc between the points $(0,0)$ and $(1,0)$. If the distance of the lemniscate point $P(x,y)$ from the origin is denoted by $r = \sqrt{x^2+y^2}$ , then this point has the coordinates $(x,y) = \left(\sqrt{\frac{1}{2}r^2(1+r^2)}, \sqrt{\frac{1}{2}r^2(1-r^2)}\right)$. The length $s(r)$ of the arc from the origin to a point at a distance *r* is then: $s(r) = \int_0^r \frac{dr}{\sqrt{1-r^4}}$



The condition that such an arc is twice as long as another can be described by the integral equation $2 \cdot \int_0^{r_1} \frac{dr}{\sqrt{1-r^4}} = \int_0^{r_2} \frac{dr}{\sqrt{1-r^4}}$. Fagnano showed that then: $r_2 = \frac{2r_1 \cdot \sqrt{1-r_1^4}}{1+r_1^4}$.

To find the point through which, for example, the arc of the lemniscate is bisected, the equation $\frac{2r \cdot \sqrt{1-r^4}}{1+r^4} = 1$ has to be solved.

Rearranging this 8th degree equation gives: $\left(\sqrt{\frac{2-\sqrt{2}}{2}}, \sqrt{\frac{3\sqrt{2}-4}{2}}\right) \approx (0.541, 0.348)$ – a point which can be constructed with compass and ruler since its coordinates are composed of square root terms.

Similar approaches result in (also constructible) decompositions of the lemniscate arc into three or five arc pieces of equal length.

Who was this Conte Fagnano e Marchese de' Toschi e di Sant'onofrio?

Giulio Carlo Fagnano came from an old patrician family in the small Adriatic town of Sinigaglia (today: Senigallia, near Ancona), which belonged to the Papal States. The parents, Giovanni Francesco Fagnani and Camilla Bartoli, noticed early on that their son was a quick learner. So he was sent to Rome to study at the *Collegio Clementino of the Padri Somaschi* (Order of the Somasci), where he was particularly interested in literature and philosophy.


After his studies, he corresponded with the philosopher Nicolas Malebranche about his book *De la recherche de la vérité*, which was later put on the index of banned books by the Catholic Church.

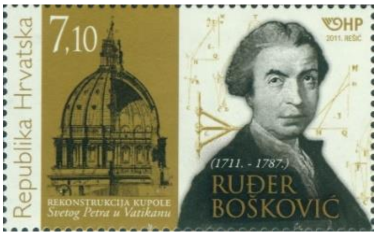
Malebranche, who, among others, was in close contact with Gottfried Wilhelm Leibniz, Johann Bernoulli and Guillaume de L'Hôpital, eventually aroused Fagnano's interest in mathematics.

In the tradition of the family, Fagnano took on responsible tasks in the administration and in the judiciary of his hometown as a *gonfaloniere* (literally: standard bearer), leaving little time for mathematical studies.


Nevertheless, he managed to bring himself up to date with the current state of mathematical research. In 1719 he derived the formula

$2 \cdot \log\left((1-\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \cdot (1+\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}\right)$ for the arc length of a quadrant – a variant of Euler's formula $e^{-\frac{\pi}{2}} = i^i$.

He received the noble title of *Count* from the French King Louis XV, that of *Marquis* from Pope Benedict XIV. Together with Ruđer Josip Bošković he advised the Pope on the cracks in the dome of St Peter's Basilica.

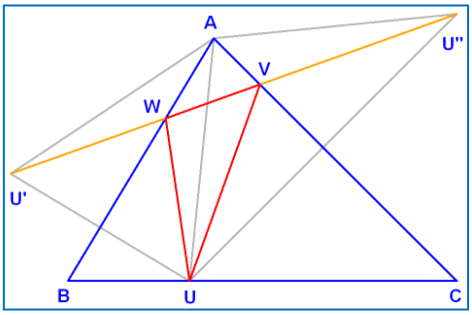



Fagnano also became known for his investigations into triangles; for example, he proved that the following equation holds for the centroid *X* of a triangle *ABC*:

$$|XA|^2 + |XB|^2 + |XC|^2 = \tfrac{1}{3} \cdot \left( |AB|^2 + |BC|^2 + |CA|^2 \right) .$$

He also dealt with an optimization problem:

> Given an acute triangle *ABC* and a point *U* on one of the sides of the triangle.
>
> *How can one construct a triangle UVW whose vertices lie on the sides of the triangle ABC and whose perimeter is minimal?*




One finds the points *V* and *W* by reflecting the segment *UA* at the sides *AB* and *AC* and connecting the mirror images of *U*, see the figure on the right.


By the way: In 1754, a young, unknown Turin student dedicated his first scientific publication to the mathematician Fagnano, whom he admired - his name: Giuseppe Luigi Lagrangia, later known as Joseph Louis Lagrange.

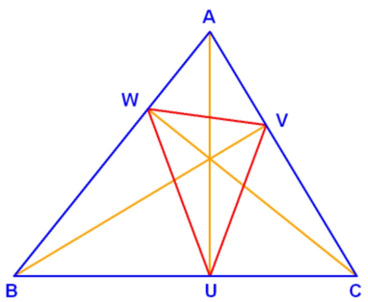
In Fagnano's marriage to Francessca Sassofrato, also from Sinigaglia, six sons and six daughters were born, of whom only a few survived.

His son Giovanni, canon of Sinigaglia Cathedral, was the only one who showed an interest in mathematics. Among other things, he published recursion formulas

for $\int x^n \sin x \, dx$ and for $\int x^n \cos x \, dx$, i.e.

$$\int x^n \sin x \, dx = -x^n \cos x + n \cdot \int x^{n-1} \cos x \, dx \quad \text{and} \quad \int x^n \cos x \, dx = x^n \sin x - n \cdot \int x^{n-1} \sin x \, dx$$

He also proved that the perimeter of an inscribed triangle in an acute-angled triangle is minimal if and only if the points *U, V, W* are the bases of the altitudes of the initial triangle, see the figure on the right.



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https://www.spektrum.de/wissen/giulio-fagnano-der-begruender-der-elliptischen-funktionen/2149284

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