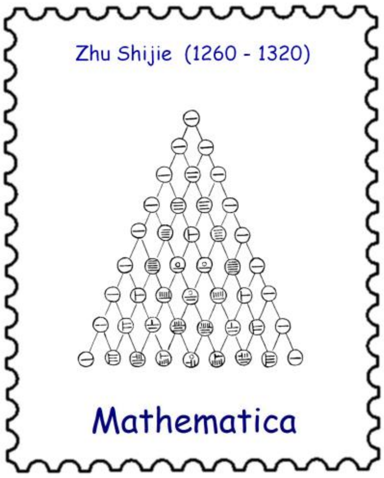
Zhu Shijie (about 1260 – about 1320)

by Heinz Klaus Strick, Germany



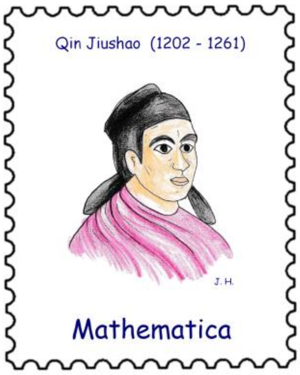
Despite China's chequered history under the rule of various dynasties, the mathematical requirements for civil service examinations remained virtually unchanged over the centuries – just as Liu Hui (220-280) had described them in his book *Jiuzhang suanshu* (Nine Chapters of Mathematical Art).

After long years of civil war-like conditions, Kublai Khan, grandson of Genghis Khan, ensured safe living conditions in China again – his armies had conquered the vast empire and built a city wall around the capital Dadu (today Beijing) in 1279.

So for more than twenty years the scholar Zhu Shijie, who was born near Dadu, could again travel through the country unhindered. His fame spread throughout the empire and many came to learn from him.

His book *Suanxue qimeng* (Introduction to Mathematical Studies, 1299), intended for beginners, went beyond the contents of the above-mentioned *Nine Chapters of Mathematical Art* in some places, for example, by also treating arithmetic with fractions and decimal fractions.

For the solution of systems of linear equations, Zhu Shijie gave recommendations on the selection of a suitable line to hold (so-called *pivoting*), and for the solution of polynomial equations he applied – like his predecessor Qin Jiushao – a method that we now call the Horner scheme.
Zhu Shijie's original work was lost, but it was reconstructed in the 19th century from a printed Korean copy from the 15th century.



The peak of mathematical development in China was reached in 1303 with Zhu Shijie's book *Siyuan yujian* (Precious Mirror of the Four Elements).

No original version of this book exists either – the version we have today contains seven prefaces, so that it is no longer known what was published by Zhu Shijie himself.

On one of the first pages of the book, one finds the diagram of the triangle that today bears Pascal's name. The binomial coefficients in the diagram correspond (in part) to the usual representation of numbers on counting sticks.

Zhu Shijie points out that he did not invent this scheme, but that this is a well-known method, presumably referring to a book by Jia Xian (1010-1070).

Using the coefficients from the last line of the figure shown, one can write down the eighth power of a sum as follows

$$(a+b)^8 = 1a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1b^8$$

Then ZHU SHIJIE goes into the treatment of tasks for which up to four variables are needed:

tian = heaven, *di* = earth, *ren* = man, *wu* = matter.

In the following, we will use the variables x, y, z, u which are common to us. The coefficients appearing in the calculations are arranged in tabular form, for example the term

	y^3	y^2	y	
	2	-8	28	
x	0	-1	6	-2
x^2	0	0	0	-1

$2y^3 - 8y^2 + 28y - xy^2 + 6xy - 2x - x^2$ is captured by the scheme on the right.

The example of the following task shows clearly how ZHU SHIJIE unfolds his method of the *celestial elements*:

- A right-angled triangle has the area 30. The sum of the lengths of the two shorter sides is 17. What is the sum of the lengths of the base (= shortest side) and the hypotenuse?

Solution: If we denote the shorter sides by x and y, and the hypotenuse by z, then we have the following: $\frac{1}{2} \cdot x \cdot y = 30 \wedge x + y = 17$.

This results in $x \cdot (17 - x) = 60$, i.e. $x^2 - 17x = -60$.

Solving the quadratic equation gives $x = 5 \vee x = 12$. The shortest side has the length 5.

The length of the hypotenuse is therefore $z = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = 13$, and it follows that the sum sought $x + z = 5 + 13 = 18$.

However, ZHU SHIJIE was not content with solving a quadratic equation. Rather, he showed how to examine the concrete problem in a more general context:

In any case, for x, y, z, PYTHAGORAS's theorem holds: $x^2 + y^2 - z^2 = 0$.

If one sets $x + z = t$ for the desired quantity, then it follows because $z = t - x$ and $y = 17 - x$, that we have: $x^2 + (17 - x)^2 - (x - t)^2 = 0$, and so $x^2 + 289 - 34x + 2xt - t^2 = 0$.

From $x^2 - 17x = -60$, i.e. $x^2 = 17x - 60$, we get a linear equation in the variable x, namely $17x - 60 + 289 - 34x + 2xt - t^2 = 0$, and so $229 - 17x + 2xt - t^2 = 0$, i.e.,

x satisfies the condition $x = \frac{229 - t^2}{17 - 2t}$.

Substituting this into the quadratic equation, we get

$\left(\frac{229 - t^2}{17 - 2t}\right)^2 + 289 - 34 \cdot \frac{229 - t^2}{17 - 2t} + 2 \cdot \frac{229 - t^2}{17 - 2t} \cdot t - t^2 = 0$ and this leads to

$$(229 - t^2)^2 + 289 \cdot (17 - 2t)^2 - 34 \cdot (229 - t^2) \cdot (17 - 2t) + 2t \cdot (229 - t^2) \cdot (17 - 2t) - t^2 \cdot (17 - 2t)^2 = 0$$

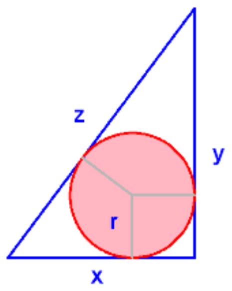
and thus finally to an equation of degree 4: $t^4 - 34t^3 + 71t^2 + 3706t + 3600 = 0$.

This equation has four solutions, namely -8, -1, 18 and 25, including the sum that is actually being searched for.

(The other solutions result in the factual context as follows: if one uses -8 as the solution for t, one gets x = 5 and z = -13; if t = -1 then x = 12 and z = -13; and t = 25 gives x = 12 and z = 13.)

Another task states:

- *The three sides x, y, z of a right triangle fulfil the conditions*
  $2yz = z^2 + xz \wedge 2x + 4y + 4z = x \cdot (y^2 - z + x)$.
  We want to solve for $u = x + y + z + d = 2x + 2y$.



The variable *d* stands for the diameter of the incircle; for this we have:

$d = 2r = x + y - z$, as already stated in *Jiuzhang suanshu*.

As a solution, ZHU SHIJIE states: $x = 3$, $y = 4$, $z = 5$ and $u = 14$.

ZHU SHIJIE traces the following problem back to a 5th degree equation:

- *The following applies to three sides x, y, z of a right triangle and the diameter d of the incircle:*
  $d \cdot x \cdot y = 24 \wedge x + z = 9$.
  What we are looking for is y.
  (Solution: $y = 3$)

ZHU SHIJIE had mastered various algebraic methods, as he demonstrated with numerous examples – including polynomials of higher degree:

To solve the quadratic equation $\boxed{-8x^2 + 578x - 3419 = 0}$, first the variable is substituted by: $x = \frac{y}{8}$.

This then gives you $-8 \cdot (\frac{y}{8})^2 + 578 \cdot \frac{y}{8} - 3419 = 0$, and so $-\frac{1}{8} y^2 + \frac{578}{8} \cdot y - 3419 = 0$ and further $-y^2 + 578 \cdot y - 27352 = 0$.

A solution to this equation is $y = 526$ and thus results in $x = \frac{526}{8} = 65\frac{3}{4}$.

With the equation $63x^2 - 740x - 432000 = 0$ he finds out that $x \approx 88$.

Shift by 88 (according to the so-called *fan fa* (celestial element) method, which we now call the HORNER method), i.e. $63 \cdot (y + 88)^2 - 740 \cdot (y + 88) - 432000 = 0$, leading to the equation $63y^2 + 10348y - 9248 = 0$, then substitution with $y = \frac{z}{63}$ gives the equation

$z^2 + 10348z - 582624 = 0$.

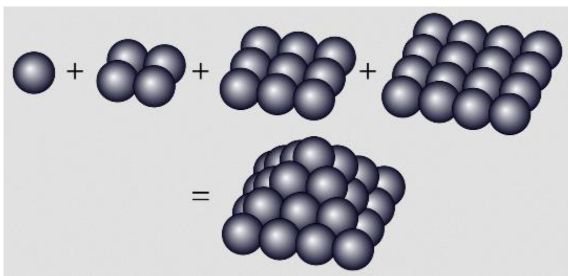
This has the solution $z = 56$, so is $y = \frac{56}{63} = \frac{8}{9}$ and thus $x = 88\frac{8}{9}$.

Early on, Chinese astronomers used approximation methods to find out the laws of planetary motion. With the help of the so-called *chao ch'a* method of continued difference formation, they determined suitable polynomials.

Example: If one forms the sum sequence of the square numbers, then one obtains the sequence of the pyramidal numbers.

Their members are therefore:

0, 1, 5, 14, 30, 55 etc.



If one forms the sequence of differences (1st order) of the sequence of pyramidal numbers, i.e., the sequence from the difference of neighbouring sequence members, then the sequence of square numbers itself results.

n	0	1	2	3	4	5
a(*n*)	0	1	4	9	16	25
s(*n*)	0	1	5	14	30	55
Δ		1	4	9	16	25
Δ²		3	5	7	9	
Δ³			2	2	2	

If one then forms further difference sequences, one arrives at a constant sequence after the third difference formation. The sum sequence $s(n)$ under consideration can therefore be described with the help of a 3rd degree polynomial: $s(n) = an^3 + bn^2 + cn + d$, where obviously $d = 0$.

The coefficients can be easily determined by taking the difference between superimposed equations:

$$\left| \begin{array}{r} a+b+c+d=1 \\ 8a+4b+2c+d=5 \\ 27a+9b+3c+d=14 \\ 64a+16b+4c+d=30 \end{array} \right| \Rightarrow \left| \begin{array}{r} 7a+3b+c=4 \\ 19a+5b+c=9 \\ 37a+7b+c=16 \end{array} \right| \Rightarrow \left| \begin{array}{r} 12a+2b=5 \\ 18a+2b=7 \end{array} \right| \Rightarrow \left| 6a=2 \right|$$

And with this (going backwards) $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$, i.e., we have:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \tfrac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1).$$

Accordingly, one can obtain the formula for their corresponding sum sequence, i.e., for the sequence 1, 6, 20, 50, ...

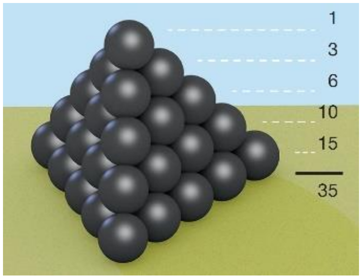
Here we obtain

$$1+5+14+30+...+\tfrac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1) = \tfrac{1}{24} \cdot n \cdot (n+1) \cdot (n+2) \cdot (2n+2)$$

and further

$$1+6+20+50+...+\tfrac{1}{24} \cdot n \cdot (n+1) \cdot (n+2) \cdot (2n+2) = \tfrac{1}{120} \cdot n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (2n+3).$$

A special role is played by the sum sequences that result from the sequence of the natural numbers, the sequence D_n of the triangular numbers (= sum sequence of the sequence of the natural numbers), the sequence T_n of the tetrahedral numbers (= sum sequence of the sequence of the triangular numbers), etc.:



Namely, if, one forms the sum sequence for the sequence $D_n = \frac{1}{2} \cdot n \cdot (n+1)$ of the triangular numbers, we get the sequence of the tetrahedral numbers 1, 4, 10, 20, 35, ..., in general a sequence with the sequence rule $T_n = \frac{1}{6} \cdot n \cdot (n+1) \cdot (n+2)$.

ZHU SHIJIE probably also discovered this sequential rule in the above-mentioned number triangle, which is still called the YANG HUI triangle in China today – after the Chinese mathematician YANG HUI (1238-1298), who also named JIA XIAN as the discoverer.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

About 350 years after ZHU SHIJIE, BLAISE PASCAL described this property, among others, in his book on the *triangle arithmétique*, which – as an example – is highlighted in colour in the figure on the left:

$$1+3+6+10=\binom{2}{0}+\binom{3}{1}+\binom{4}{2}+\binom{5}{3}=\binom{6}{3}=\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}=20,$$

i.e., the sum of the first four triangular numbers is 20. In general:

$$1+3+6+10+\ldots+\tfrac{1}{2!}\cdot n\cdot(n+1)=\binom{2}{0}+\binom{3}{1}+\binom{4}{2}+\ldots+\binom{n+1}{n-1}=\binom{n+2}{n-1}=\tfrac{1}{3!}\cdot n\cdot(n+1)\cdot(n+2)$$

Accordingly, one finds the corresponding sum sequence for this, i.e., the sum sequence with the members 1, 5, 15, 35, 70, ..., by looking at the next line in the PASCAL triangle:

$1+4+10+20+\ldots+\tfrac{1}{3!}\cdot n\cdot(n+1)\cdot(n+2)=\tfrac{1}{4!}\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)$ and further

$$1+5+15+35+\ldots+\tfrac{1}{4!}\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)=\tfrac{1}{5!}\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4),$$

$$1+6+21+56+\ldots+\tfrac{1}{5!}\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)=\tfrac{1}{6!}\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot(n+5).$$

First published 2022 by Spektrum der Wissenschaft Verlagsgesellschaft Heidelberg

https://www.spektrum.de/wissen/der-ruhmreiche-zhu-shijie/2065983

Translated 2022 by John O'Connor, University of St Andrews
