A transcription of Tait's notes on Terrot's lecture

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On the
Imaginary roots of Negative Quantities.
By the Right Reverend Bishop Terrot.
1847

1. $\sqrt{-1}$ is called impossible or imaginary $\because$ no ordinary algebraic quantity which must be either $+$ or $-$ can give when squared a negative result. Considering however the common application of Algebra to Geometry we easily see, that the assumption that every line must be either $+$ or $-$ is inconsistent with the possibility of drawing a line in any direction. $+1 \times a$ means a line whose length is a drawn in one direction, $-1 \times a$ means the same length of line but drawn in a different direction, and to say that a line of the length of a cannot be drawn in any other direction than one of these is absurd. $\sqrt{-1}$ $\therefore$ is not impossible any more than $-$ or $+1$ and shows only the direction of the line to which it is affixed.

2. If from C [See Fig 1, Lewis] we draw any number of lines such that they shall be in continued proportion and make at the same time $\angle ACA_1 = A_1CA_2 = A_2CA_3$ &c then calling $CA = 1$, $CA_1 = a$, $CA_2 = a^2$ or the lines are in this series a^0, a^1, a^2, a^3 &c while the angles which they make with the line CA are $0, \vartheta, 2\vartheta, 3\vartheta$ &c being the angle $ACA_1\times$ exponent of that radius vector (CA_a for example) from which to CA they are measured. Thus the line whose angle of inclination is on $n\vartheta$ has its length $= a^n$ & vice versâ.

3. If we now assume the several lines CA, CA_1, CA_2, &c [See Fig 2, Lewis] all equal or radii of a circle the case will not be altered. Let n be a divisor of $2r\pi$ or let $\vartheta = \frac{2r\pi}{n}$. Thus the Radius $a^n = a^{\frac{2r\pi}{\vartheta}}$ is the same in length & position as CA $\therefore a^1 = 1^{\frac{1}{n}} = 1^{\frac{\vartheta}{2r\pi}}$. We know from ordinary Algebraical principles that the several nth roots of unity may be expressed by the series a, a^2, a^3, &c. It therefore follows that we may take the successive Radii of a circle at equal angles for the several roots of unity & conversely. If R be the numerical length of radius that radius inclined to the first at $\angle\vartheta$ is $= R \times 1^{\frac{\vartheta}{2r\pi}}$. We $\therefore$ call $1^{\frac{\vartheta}{2r\pi}}$ the coefficient of direction because it refers only to the direction, never to the length of a line. Thus, $a \times \frac{1+\sqrt{-3}}{2}$ is a line $= a$ simply.

4. Let us next suppose $n = 2$, AB will be a diameter & if $CA = 1$, $CB = -1$. But $a^2 = 1 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ & $a^2 = +1$.

Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But the roots are ± 1 & $\pm\sqrt{-1}$. Here CA & CB are symbolized by $+1$ & -1 respectively $\therefore$ CD & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive or negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing Props. that if a line is symbolised by $= a \cdot 1^{\frac{\vartheta}{2r\pi}}$ we know both its length & direction. $a \cdot 1^{\frac{\vartheta}{2r\pi}}$ $\therefore$ represents the actual transference of the point in space by moving from A to C. [See Fig 3, Lewis] But it is also clear that its actual transference in space though not its distance travelled would be the same did it move from A to B & then from B to C. Thus $\therefore$ ($AC\times$ its coefficient of direction) $=$ ($AB\times$ its coefficient of direction) $+$ ($BC\times$ its coefficient of direction). Therefore also the sum of any two lines making an angle with each other is $=$ the diagonal of their parallelogram completed. Even in this startling form it is only the general assertion of a proposition particular cases of which we admit when we say $AB_1 + B_1C = AC$ or that $AC + CB_1 = AB_1$.

1. As examples to elucidate this let ABC (Fig 4) [See Fig 3, Lewis] be an isosceles right angled triangle described on the radius AD. If we call AB the radius or Hypotenuse a each of the sides will be in length $\frac{a}{\sqrt{2}}$ & AB is symbolized by $a\times 1^{\frac{45}{360}} = a\times 1^{\frac{1}{8}} = a\times\frac{1+\sqrt{-1}}{\sqrt{2}}$. But $AC = \frac{a}{\sqrt{2}}$. CB being perpendicular to original position is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop. 4) $\therefore$ $AC + CB = a \times \left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.

2. Let $BAC = 60°$, $BCA = 90°$, then AB in length & direction is $a \cdot 1^{\frac{60}{360}} = a \cdot 1^{\frac{1}{6}} = a\cdot\frac{1+\sqrt{-3}}{2}$, $AC = \frac{a}{2}$, CB in length $= a\cdot\frac{\sqrt{3}}{2}$ $\therefore$ in length & direction jointly $= a\cdot\frac{\sqrt{3}\sqrt{-1}}{2} = a\cdot\frac{\sqrt{-3}}{2}$ $\therefore AC + CB = \frac{a}{2} + a\cdot\frac{\sqrt{-3}}{2} = a\cdot\frac{1+\sqrt{-3}}{2} = AB$.

3. Let the triangle (Fig 5) [See Fig 3, Lewis] be Equilateral & let AB be the original position. Let $AB = a$, $AC = a \cdot 1^{\frac{1}{6}}$, $CB = a \cdot 1^{\frac{-1}{6}}$ $\therefore AC + CB = a \cdot \left[1^{\frac{1}{6}} + 1^{\frac{-1}{6}}\right] = a \cdot \left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{-1+\sqrt{-3}}{2} + 1\right] \times \frac{2}{1+\sqrt{-3}} = a \cdot \left[\frac{1+\sqrt{-3}}{2} + \frac{2}{1+\sqrt{-3}}\right] = a = AB$

6. In the foregoing Props. & Examples it has been taken for granted that we know not only the several nth roots of unity but also their proper order; that is the order in which as coefficients they express the radii drawn to the extremities of the arcs $\vartheta, 2\vartheta, 3\vartheta$, &c. with the original radius. But when we determine the roots of $x^n - 1 = 0$ we obtain them in no fixed order. To discover this order we must observe that two roots are always of the form $a \pm \sqrt{-b}$ comparing which with (Fig 6) [See Fig 4, Lewis] a is evidently the part symbolical of the cosine $+\sqrt{-b}$ that of the sine because it is affected by $\sqrt{-1}$ and is $\therefore$ perpendicular to original radius. Thus $\therefore$ in $a \pm \sqrt{-b}$,$+$ refers to radii in the upper semicircle & $-$ to those in the under; and the two radii whose symbols differ only in the sign of $\sqrt{-b}$ are at equal angles to the original radius on opposite sides of it. $\therefore$ the root in which a is greatest is nearest to the original radius. Thus the roots of $n^6 - 1$ arranged properly are

$$1, \frac{1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2}, -1, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$$

symbolizing the radii drawn respectively to the ends of the arcs

$$0° \text{ or } 360°, 60°, 120°, 180°, 240°, 300°$$

For if $+1$ be first -1 having no sinal part must be in the middle. Next $\frac{1+\sqrt{-3}}{2}$ & $\frac{-1+\sqrt{-3}}{2}$ must be in the upper half of the circle and $\frac{1+\sqrt{-3}}{2}$ must come first because its cosine is in CA. And so with the rest.

7. It appears from Props. 4, 5 that the radius drawn to the end of an arc ϑ is $= 1^{\frac{\vartheta}{2r\pi}}$ and this again by $a \pm \sqrt{-b}$ where a is what is trigonometrically called the cosine & $\sqrt{b}$ the sine of ϑ. Now (Fig 6) [See Fig 4, Lewis] let $\angle ACA_1 = \vartheta$, $\angle ACA_2 = 2\vartheta$, &c $\angle ACA_p = p\vartheta$, then

$$CA_1 = CD + \sqrt{-1} \cdot DA_1 = \cos\vartheta + \sqrt{-1} \cdot \sin\vartheta,$$
$$CA_p = \cos p\vartheta + \sqrt{-1} \cdot \sin p\vartheta$$

But by prop. 2,

$$CA_p = \overline{CA_1}^p = \left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)^p$$
$$\therefore \left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)^p = \cos p\vartheta + \sqrt{-1} \sin p\vartheta, \text{ which is}$$

Demoivre's Theorem.

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cor. If $p\vartheta = 2\pi$, $\cos p\vartheta + \sqrt{-1} \cdot \sin p\vartheta = 1$.
Hence $\left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)$, $\left(\cos 2\vartheta + \sqrt{-1} \cdot \sin 2\vartheta\right)$ &c. represent the several pth roots of unity. If we arrange the angles, instead of $\vartheta, 2\vartheta, 3\vartheta$ &c, in pairs thus ϑ & $\overline{p-1} \cdot \vartheta$, 2ϑ & $\overline{p-2} \cdot \vartheta$ &c. the several expressions for x—the several pth roots of unity or the simple factors of $x^p - 1 = 0$ taken in pairs corresponding with the above will be

$$\left(x - \cos\vartheta - \sqrt{-1} \cdot \sin\vartheta\right) \& \left(x - \cos\overline{p-1}\,\vartheta - \sqrt{-1} \cdot \sin\overline{p-1}\,\vartheta\right)$$

which last is $= \left(x - \cos\overline{p\vartheta - \vartheta} - \sqrt{-1} \cdot \sin\overline{p\vartheta - \vartheta}\right) =$
$\left(x - \cos\overline{2\pi - \vartheta} - \sqrt{-1} \cdot \sin\overline{2\pi - \vartheta}\right) = \left(x - \cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)$

In the same way the next pair must be

$$\left(x - \cos 2\vartheta + \sqrt{-1} \cdot \sin 2\vartheta\right) \& \left(x - \cos 2\vartheta - \sqrt{-1} \cdot \sin 2\vartheta\right)$$

Multiplying these together for the quadratic factors of $x^p - 1$, we obtain when p is even

$$x^p - 1 = (x^2 - 1)(x^2 - 2x\cos\vartheta + 1)\cdot(x^2 - 2x\cos 2\vartheta + 1) \text{ to } \tfrac{p}{2} \text{ terms}$$

But when p is odd

$$x^p - 1 = (x - 1)(x^2 - 2x\cos\vartheta + 1) \text{ \&c to } \tfrac{p+1}{2} \text{ terms}$$

where ϑ it may be observed is $= \frac{2\pi}{p}$

8.

$$\sin\overline{A+B} = \sin A\cdot\cos B + \cos A\cdot\sin B.$$

$$\cos\overline{A+B} = \cos A\cdot\cos B - \sin A\cdot\sin B.$$

Let arc AB (Fig 7) [See Fig 5, Lewis] $= \mathcal{A}$, BD_2 & AD_1 each $= \mathcal{B}$.

Then by Prop. 3, $CB = r\cdot 1^{\frac{\mathcal{A}}{2\pi}}, CD_1 = r\cdot 1^{\frac{\mathcal{B}}{2\pi}}, CD_2 = r\cdot 1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}}$

$\therefore CD_2 = r\cdot 1^{\frac{\mathcal{A}}{2\pi}}\cdot 1^{\frac{\mathcal{B}}{2\pi}}$

But by Prop. 7,

$1^{\frac{\mathcal{A}}{2\pi}} = \cos\mathcal{A} + \sqrt{-1}\cdot\sin\mathcal{A}$

$1^{\frac{\mathcal{B}}{2\pi}} = \cos\mathcal{B} + \sqrt{-1}\cdot\sin\mathcal{B}$

$\therefore 1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}} = \cos\mathcal{A}\times\cos\mathcal{B} - \sin\mathcal{A}\times\sin\mathcal{B} + \sqrt{-1}\Big(\sin\mathcal{A}\cdot\cos\mathcal{B} + \cos\mathcal{A}\cdot\sin\mathcal{B}\Big),$

but $1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}} = \cos\overline{\mathcal{A}+\mathcal{B}} + \sqrt{-1}\sin\overline{\mathcal{A}+\mathcal{B}}.$

Equating then the sinal and cosinal parts of these, we have,

$$\cos\mathcal{A}\cdot\cos\mathcal{B} - \sin\mathcal{A}\cdot\sin\mathcal{B} = \cos\overline{\mathcal{A}+\mathcal{B}}$$

$$\sin\mathcal{A}\cdot\cos\mathcal{B} + \cos\mathcal{A}\cdot\sin\mathcal{B} = \sin\overline{\mathcal{A}+\mathcal{B}}$$

Definition

It should be observed that in the following propositions a line expressed by letter simply as AB must be considered both as to length & direction while when in brackets thus (AB) its length alone is referred to. Thus $(AB)1^{\frac{\vartheta}{2\pi}} = AB$.

9. In any right angled triangle the sum of the squares of the sides is = square of hypotenuse.

Let CA (Fig 6) [See Fig 4, Lewis] $= r$, then $CA_1 = r \cdot 1^{\frac{\vartheta}{2\pi}}$, & $CA_{n-1} = r \cdot 1^{\frac{-\vartheta}{2\pi}}$

$\therefore CA_1 \times CA_{n-1} = r^2 \times 1^{\frac{\vartheta}{2\pi}} \times \frac{1}{1^{\frac{\vartheta}{2\pi}}} = r^2,$

Also $CA_1 = (CD_1) + \sqrt{-1}(D_1A_1)$

$CA_{n-1} = (CD_1) - \sqrt{-1}(D_1A_1)$ for $(D_1A_1) = (D_1A_{n-1})$

$\therefore CA_1 \times CA_{n-1} = (CD_1)^2 + (D_1A_1)^2$ which is $\therefore = r^2 = (CA)^2 = (CA_1)^2$

its equivalent in area.

10. ***Cotes' Properties of the Circle.***

Let the circumference be divided into n equal parts and join OP_1, OP_2, OP_3, &c (Fig 8) [See Fig 6, Lewis] and also join P_1, P_2, P_3 with C any point in the Diameter. Then

$CP_1 = OP_1 - OC$, $CP_2 = OP_2 - OC$ &c

$\therefore CP_1 \cdot CP_2 \cdot CP_3 \cdots CP_n = \Sigma_n \cdot (OA)^n - \Sigma_{n-1} \cdot (OA)^{n-1} \ldots\ldots \pm OC^n,$

where Σ_n is the product of all the coefficients of direction for OP_1, OP_2, &c, Σ_{n-1} the sum of $\wedge$ (*the product sq? P.G. Tait*) these coefficients taken $\overline{n-1}$ together & so on. But these coefficients are also the roots of the Equation $x^n - 1 = 0$. Now the product of the roots of this Equation with their signs changed is -1 & Σ_n is = the product with their signs unchanged. Therefore if n be even $\Sigma_n = -1$ but if odd $+1$, and in either case Σ_{n-1}, Σ_{n-2} &c each = 0. Hence $CP_1 \cdot CP_2 \cdot CP_3 \cdots CP_n = \pm(OA)^n \pm (OC)^n$; the upper signs to be used when n is even, the lower when odd.

Here CP_1, CP_2 &c consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all their coefficients of direction. If n be even the several pairs as CP_1, CP_{n-1} are evidently of the form $(CP_1)\cdot 1^{\frac{\vartheta}{2\pi}}$ and $(CP_{n-1})\cdot 1^{\frac{-\vartheta}{2\pi}}$ $\therefore CP_1 \times CP_{n-1} = (CP_1) \times (CP_{n-1})$ and this is true for every pair except $CA = (CA) \cdot +1$ & $CB = (CB) \cdot -1 \therefore (CP_1) \cdot (CP_2) \cdots (CP_n) = (-OA_n^n + OC^n) \cdot -1 = OA^n - OC^n$

But if n be odd the several pairs remain as before only no P falling on B, -1 is not a coefficient of direction $\therefore (CP_1) \cdot (CP_2) \cdot$ &c $= OA^n - OC^n$ as before.

Cor.1. If C be on the opposite side of O from A, the other conditions remaining the same OC is negative. If n be even the deduction in the prop. remains unchanged. But if n be odd, $(CP_1) \cdot (CP_2) \cdot \cdot$ &c $= OA^n + OC^n$. Here it may be remarked that when lines as OA are in the original direction, since the coefficient of direction in that case is unity it is immaterial whether we write OA or (OA).

Ex. Let $n = 3$ & $OC = \frac{1}{2}$,

then, $(AC) = \frac{3}{2}$, $(CP_1) = (CP_2) = \frac{\sqrt{3}}{2}$

$$\therefore (CA) \cdot (CP_1) \cdot (CP_2) = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{9}{8} = 1 + \frac{1}{8} = \overline{1}^3 + \overline{\tfrac{1}{2}}^3 = OA^3 + OC^3.$$

Cor.2. If C be in OA produced the reasoning & result will be the same as in the prop., only, that now CA & CB being of the same affection -1 is not a divisor of the second member of the Equation, &,

$$(CP_1) \cdot (CP_2) \cdot \text{ \&c} = (OC)^n - (OA)^n.$$

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11. If from A the extremity of the Diameter (Fig 8) [See Fig 6, Lewis] the circumference be divided into n equal parts & if these several extremities be joined, then

$$(AP_1) \cdot (AP_2)(AP_{n-1}) = nCA^{n-1}$$

As in former prop. $AP_1 = CP_1 - CA$, $AP_2 = CP_2 - CA$ & so on

$$\therefore AP_1 \cdot AP_2 \cdot \cdot AP_{n-1} = \overline{CP_1 - CA} \cdot \overline{CP_2 - CA} \text{ \&c to } \overline{n-1} \text{ factors}$$

$$= R^{n-1} \cdot \{S_{n-1} - S_{n-2} \dots . \pm S_1 \pm 1\}$$

where S_1, S_2 &c are the sum, sum of products two & two, &c of all the values of $1^{\frac{1}{n}}$ except unity there being no line drawn from A to the circumference in the direction CA. S_1, S_2 &c are $\therefore$ the coefficients of the Equation $\frac{x^n-1}{x-1}$ or of $x^{n-1} + x^{n-2} + \dots$ &c with the signs changed for the products of odd ascending roots, unchanged for even ones.

If $\therefore$ $\overline{n-1}$ be even $S_{n-1} = +1$, $S_{n-2} = -1$, & so on,

if $\overline{n-1}$ be odd $S_{n-1} = -1$, $S_{n-2} = +1$ & so on.

$\therefore AP_1 \cdot AP_2 \cdot$ &c $= R^{n-1} \times \pm\{1 + 1 + 1$ to n terms$\} = \pm nR^{n-1}$ according as $\overline{n-1}$ is even or odd.

If $\overline{n-1}$ be even, $AP_1 \cdot AP_2 \cdot$ &c $= (AP_1)(AP_2)$ &c $\cdot$ the several pairs of coefficients giving unity for their products.

If $\overline{n-1}$ be odd, then the several pairs give as before their product unity but there remains the factor AB which has for its coefficient -1.

$\therefore$ in either case $(AP_1)(AP_2)\&c(AP_{n-1}) = nR^{n-1}$

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12. If by this method we undertake to prove that the angles at the base of an Isosceles triangle are = eachother we have $(AC) = (BC)$ (Fig 5). [See Fig 3, Lewis]

But $AC = (AC) \cdot 1^{\frac{A}{2\pi}} = (AC) \cdot [a + \sqrt{-b}]$,

$CB = AD = (AC) \cdot 1^{\frac{-B}{2\pi}} = (AC) \cdot [a' + \sqrt{-b'}]$.

But $AC + CB = AB$.

$\therefore (AC) \cdot (a + a' + \sqrt{-b} + \sqrt{-b'}) = AB =$ a positive quantity

consequently the sinal parts destroy one another or $\sqrt{-b} = -\sqrt{-b'}$ or $b = -b'$. Therefore the angles A & B have their sines of equal length but of different affections. The angles themselves $\therefore$ being together less than π are geometrically equal to each other.

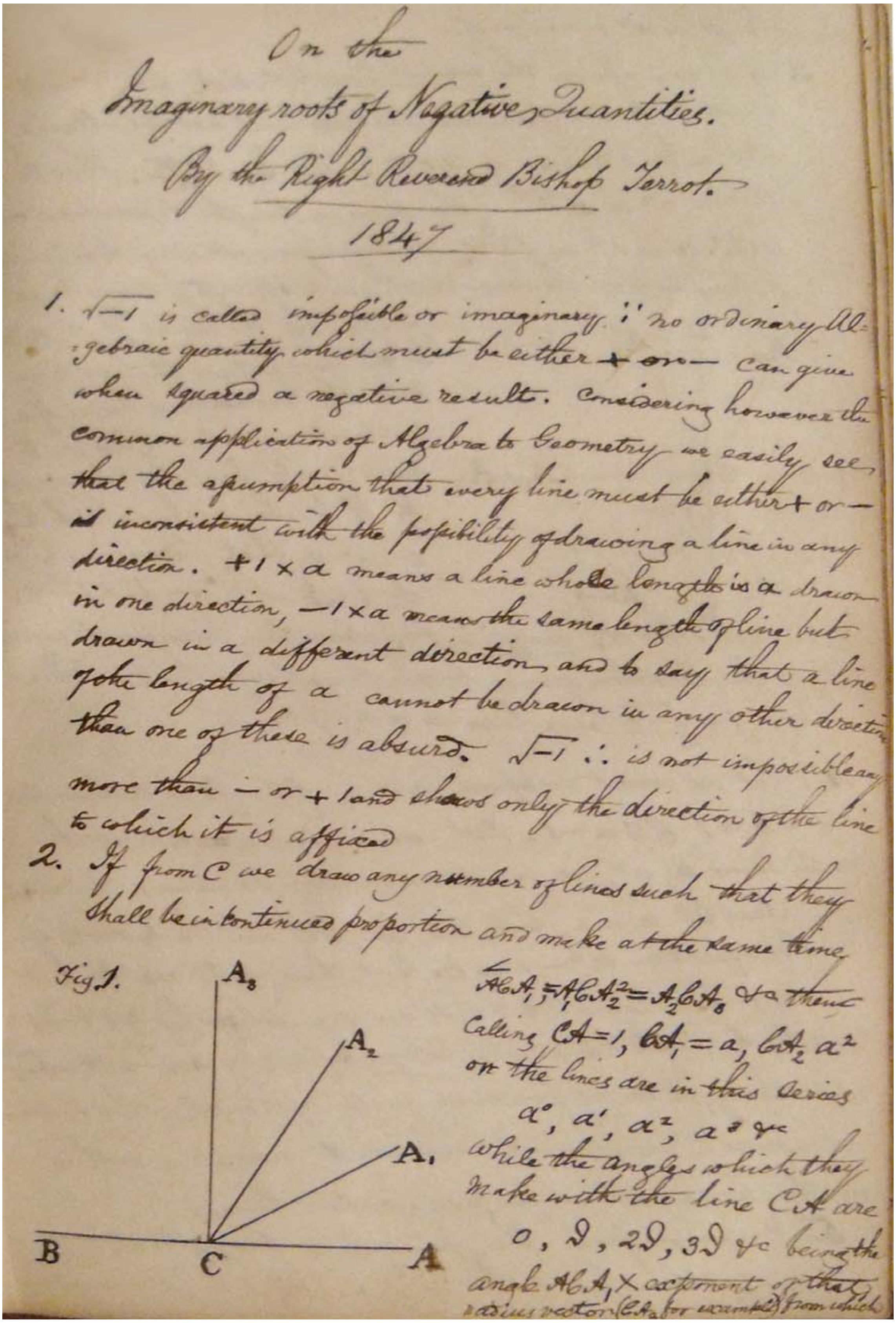
Cor. Much in the same way we might prove that in every triangle the greater side has the greater angle opposite to it & vice versâ that the greater angle has the greater side opposite to it.

May 27th 1847
P.G. Tait.

Appendices

A Images to accompany the text: Figures 1–8

Please note that my Figure 3 contains Figures 3, 4 and 5 as referred to in the text, which throws out the subsequent correspondence between Tait's numbering of figures and mine. I feel this is a necessary inconvenience as it allows the reader to view the figures in context. Readers are directed to the appropriate figure by comments within square brackets, e.g. [See Fig 3, Lewis].



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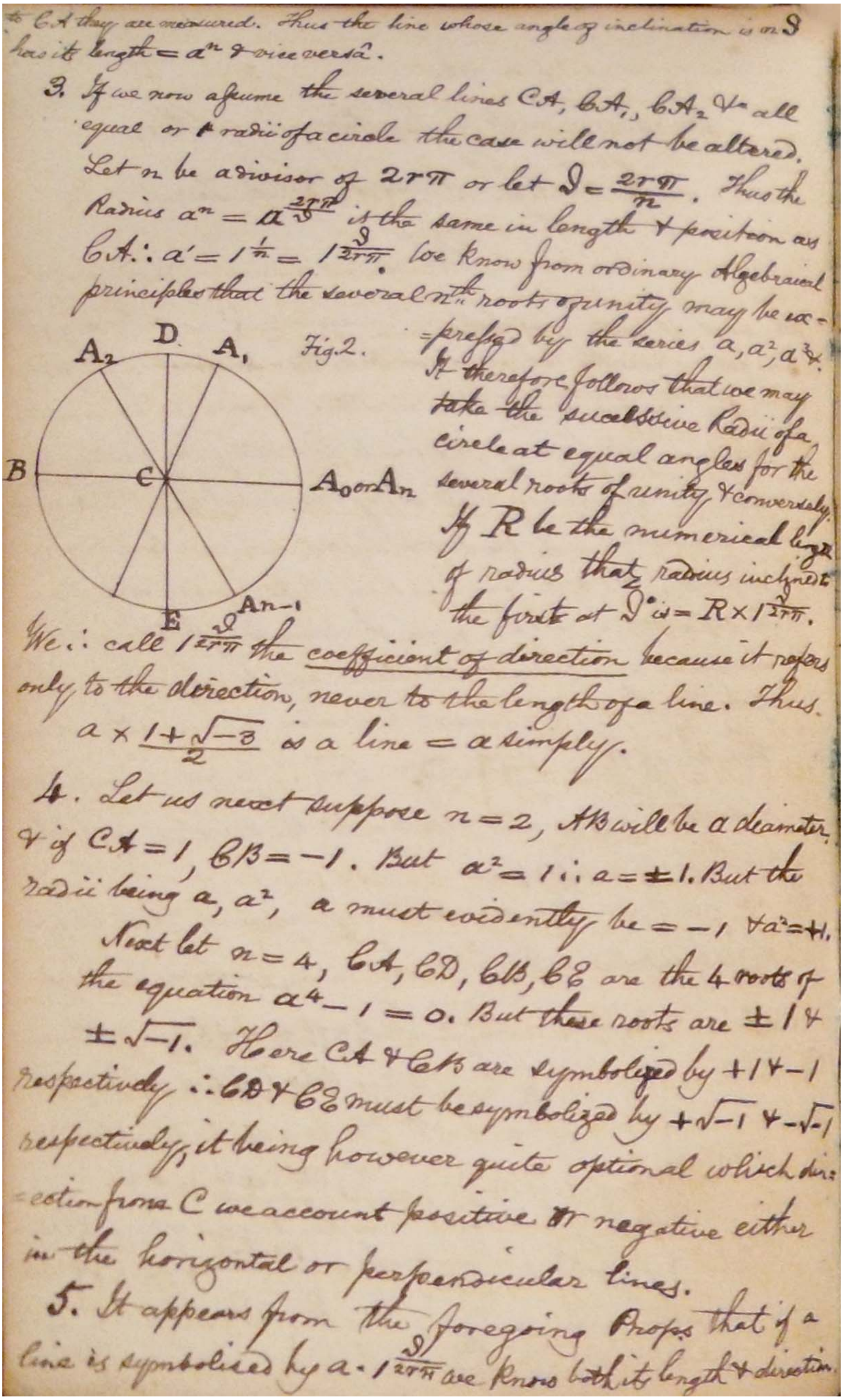
2. If from C we draw any number of lines such that they shall be in continued proportion and make at the same time

Fig 1.

$\angle ACA_1 = A_1CA_2 = A_2CA_3$ &c then
Calling $CA = 1$, $CA_1 = a$, CA_2 a^2
or the lines are in this series
a^0, a^1, a^2, a^3 &c
while the angles which they make with the line CA are
$0, \theta, 2\theta, 3\theta$ &c being the angle $ACA_1 \times$ exponent of that radius vector (CA_2 for example) from which

Figure 1: An extract from Terrot's lecture; Tait's drawing of Figure 1 (Tait–Maxwell School-book)



to CA they are measured. Thus the line whose angle of inclination is $n\vartheta$ has its length $= a^n$ & vice versâ.

3. If we now assume the several lines CA, CA_1, CA_2 &c all equal or radii of a circle the case will not be altered. Let n be a divisor of $2r\pi$ or let $\vartheta = \frac{2r\pi}{n}$. Thus the Radius $a^n = a^{\frac{2r\pi}{\vartheta}}$ is the same in length & position as CA: $\therefore a' = 1^{\frac{1}{n}} = 1^{\frac{\vartheta}{2r\pi}}$. We know from ordinary Algebraical principles that the several n^{th} roots of unity may be expressed by the series a, a^2, a^3 &c. It therefore follows that we may take the successive Radii of a circle at equal angles for the several roots of unity & conversely. If R be the numerical length of radius that radius inclined to the first at $\vartheta°$ is $= R \times 1^{\frac{\vartheta}{2r\pi}}$.

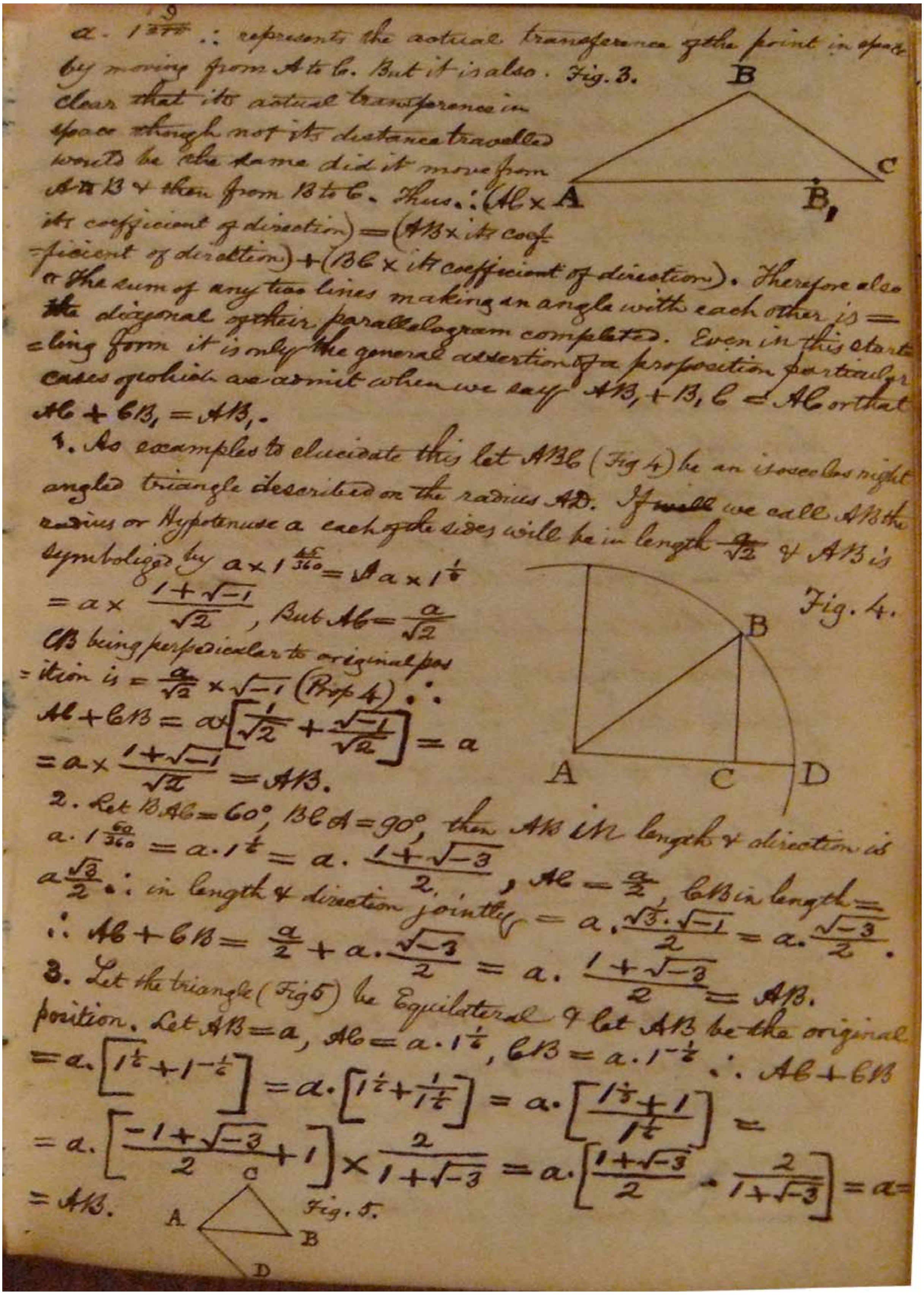
We $\therefore$ call $1^{\frac{\vartheta}{2r\pi}}$ the coefficient of direction because it refers only to the direction, never to the length of a line. Thus $a \times \frac{1 + \sqrt{-3}}{2}$ is a line $= a$ simply.

4. Let us next suppose $n = 2$, AB will be a diameter, & if $CA = 1$, $CB = -1$. But $a^2 = 1 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ & $a^2 = +1$.

Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But these roots are ± 1 & $\pm\sqrt{-1}$. Here CA & CB are symbolized by $+1$ & -1 respectively $\therefore$ CD & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive or negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing Props that if a line is symbolised by $a \cdot 1^{\frac{\vartheta}{2r\pi}}$ we know both its length & direction.

Figure 2: An extract from Terrot's lecture; Tait's drawing of Figure 2 (Tait–Maxwell School-book)



$a \cdot 1^{[illegible]}$ ∴ represents the actual transference of the point in space by moving from A to C. But it is also clear that its actual transference in space though not its distance travelled would be the same did it move from A to B & then from B to C. Thus ∴ (AC × its coefficient of direction) = (AB × its coefficient of direction) + (BC × its coefficient of direction). Therefore also or the sum of any two lines making an angle with each other is = the diagonal of their parallelogram completed. Even in this startling form it is only the general assertion of a proposition particular cases of which we admit when we say $AB_1 + B_1C = AC$ or that $AC + CB_1 = AB_1$.

Fig. 3.

1. As examples to elucidate this let ABC (Fig 4) be an isosceles right angled triangle described on the radius AD. If we call AB the radius or Hypotenuse a each of the sides will be in length $\frac{a}{\sqrt{2}}$ & AB is symbolized by $a \times 1^{\frac{45}{360}} = a \times 1^{\frac{1}{8}} = a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$, But $AC = \frac{a}{\sqrt{2}}$ CB being perpendicular to original position is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop 4) ∴ $AC + CB = a\left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.

Fig. 4.

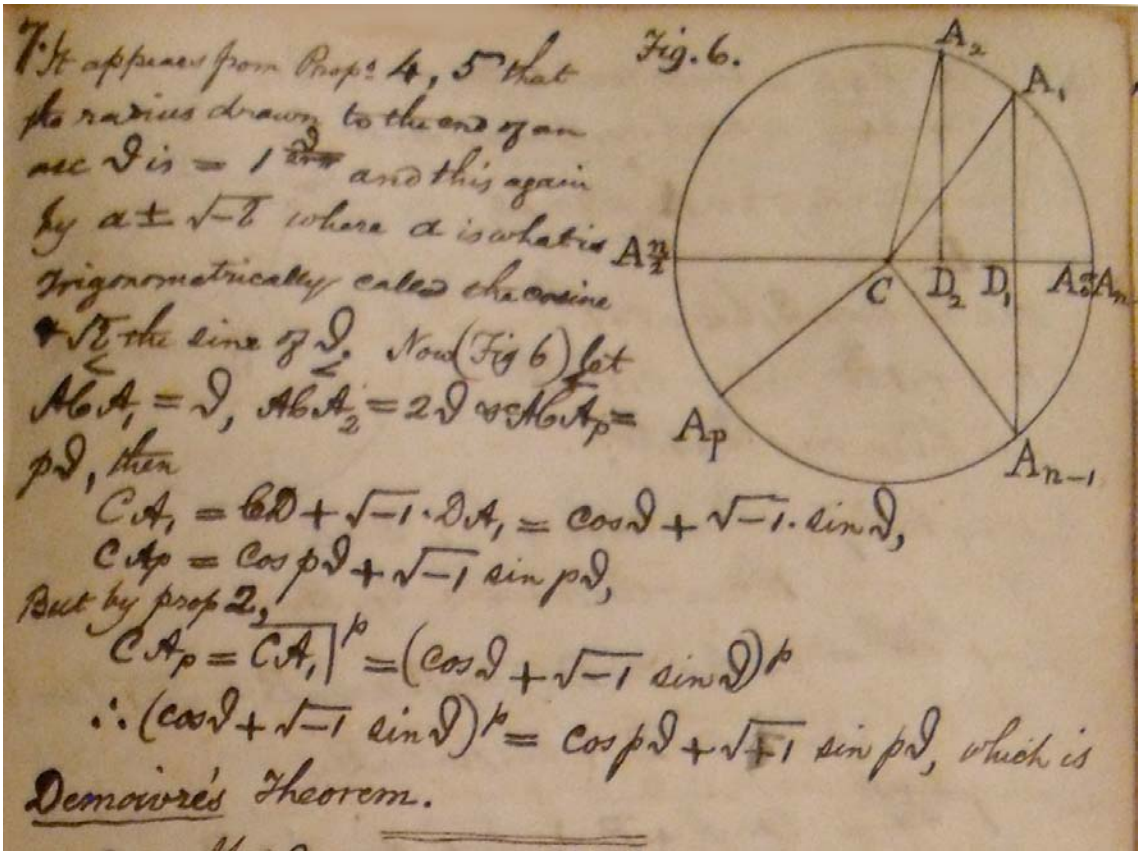
2. Let $BAC = 60°$, $BCA = 90°$, then AB in length & direction is $a \cdot 1^{\frac{60}{360}} = a \cdot 1^{\frac{1}{6}} = a \cdot \frac{1+\sqrt{-3}}{2}$, $AC = \frac{a}{2}$, CB in length $= a\frac{\sqrt{3}}{2}$ ∴ in length & direction jointly $= a \cdot \frac{\sqrt{3} \cdot \sqrt{-1}}{2} = a \cdot \frac{\sqrt{-3}}{2}$.

∴ $AC + CB = \frac{a}{2} + a \cdot \frac{\sqrt{-3}}{2} = a \cdot \frac{1+\sqrt{-3}}{2} = AB$.

3. Let the triangle (Fig 5) be Equilateral & let AB be the original position. Let $AB = a$, $AC = a \cdot 1^{\frac{1}{6}}$, $CB = a \cdot 1^{-\frac{1}{6}}$ ∴ $AC + CB = a\left[1^{\frac{1}{6}} + 1^{-\frac{1}{6}}\right] = a \cdot \left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{-1+\sqrt{-3}}{2} + 1\right] \times \frac{2}{1+\sqrt{-3}} = a \cdot \left[\frac{1+\sqrt{-3}}{2} \cdot \frac{2}{1+\sqrt{-3}}\right] = a = AB$.

Fig. 5.

Figure 3: An extract from Terrot's lecture; Tait's drawing of Figures 3, 4, 5 (Tait–Maxwell School-book)



7. It appears from Propˢ 4, 5 that the radius drawn to the end of an arc ϑ is $= 1^{\frac{\vartheta}{2\pi}}$ and this again by $a \pm \sqrt{-b}$ where a is what is trigonometrically called the cosine & $\sqrt{b}$ the sine of ϑ. Now (Fig 6) let $ACA_1 = \vartheta$, $ACA_2 = 2\vartheta$ &c $ACA_p = p\vartheta$, then

$$CA_1 = CD + \sqrt{-1}\cdot DA_1 = \cos\vartheta + \sqrt{-1}\cdot\sin\vartheta,$$

$$CA_p = \cos p\vartheta + \sqrt{-1}\sin p\vartheta,$$

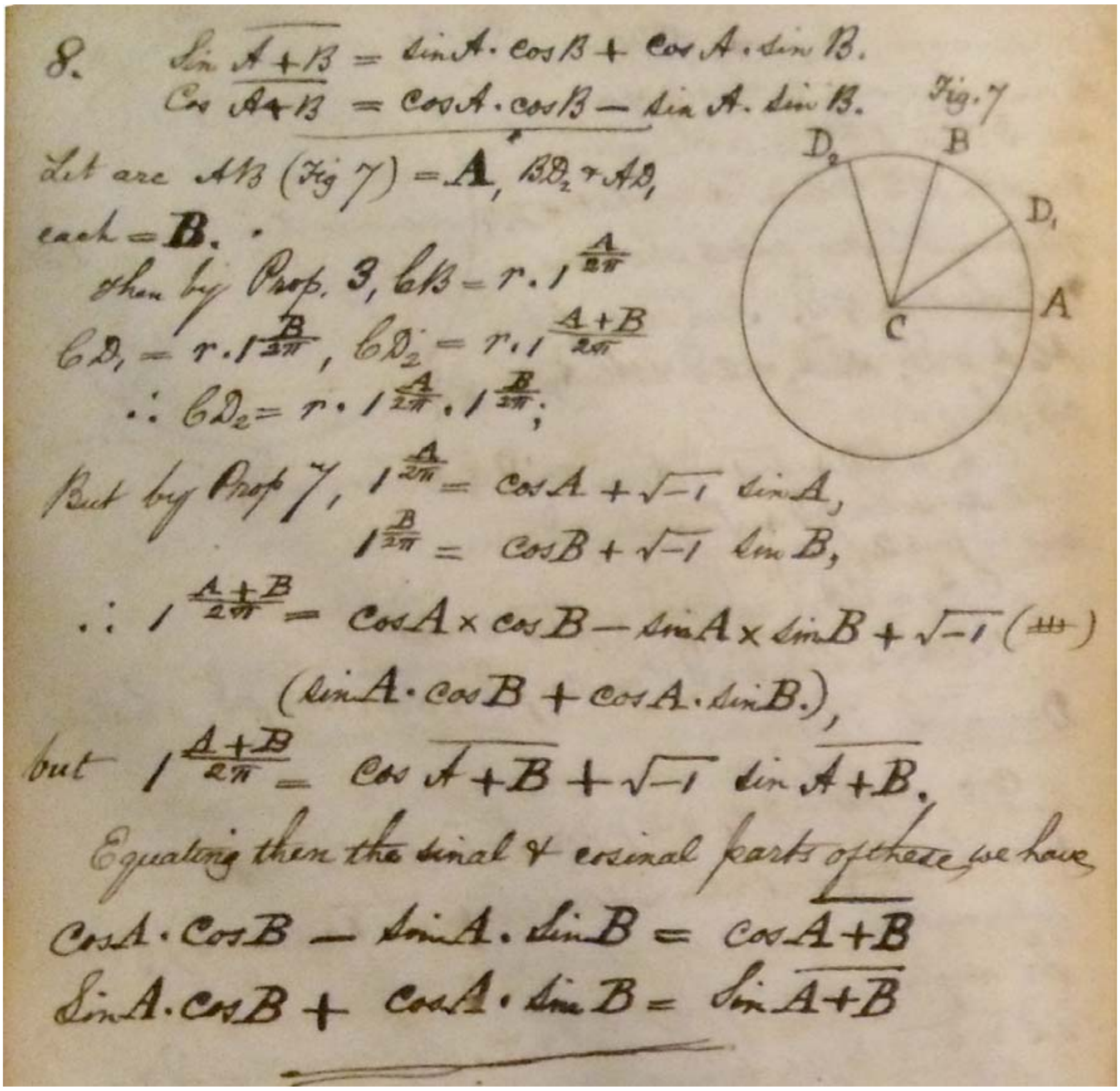
But by prop 2,

$$CA_p = \overline{CA_1}|^p = (\cos\vartheta + \sqrt{-1}\sin\vartheta)^p$$

$$\therefore (\cos\vartheta + \sqrt{-1}\sin\vartheta)^p = \cos p\vartheta + \sqrt{-1}\sin p\vartheta,$$ which is

Demoivre's Theorem.

Figure 4: An extract from Terrot's lecture; Tait's drawing of Figure 6 (Tait–Maxwell School-book)



8. $\sin \overline{A+B} = \sin A \cdot \cos B + \cos A \cdot \sin B.$

$\cos \overline{A+B} = \cos A \cdot \cos B - \sin A \cdot \sin B.$ Fig. 7

Let arc AB (Fig 7) = **A**, BD_2 & AD_1 each = **B**.

then by Prop. 3, $CB = r \cdot 1^{\frac{A}{2\pi}}$

$CD_1 = r \cdot 1^{\frac{B}{2\pi}}$, $CD_2 = r \cdot 1^{\frac{A+B}{2\pi}}$

$\therefore CD_2 = r \cdot 1^{\frac{A}{2\pi}} \cdot 1^{\frac{B}{2\pi}};$

But by Prop 7, $1^{\frac{A}{2\pi}} = \cos A + \sqrt{-1} \sin A,$

$1^{\frac{B}{2\pi}} = \cos B + \sqrt{-1} \sin B,$

$\therefore 1^{\frac{A+B}{2\pi}} = \cos A \times \cos B - \sin A \times \sin B + \sqrt{-1}\,(\sin A \cdot \cos B + \cos A \cdot \sin B),$

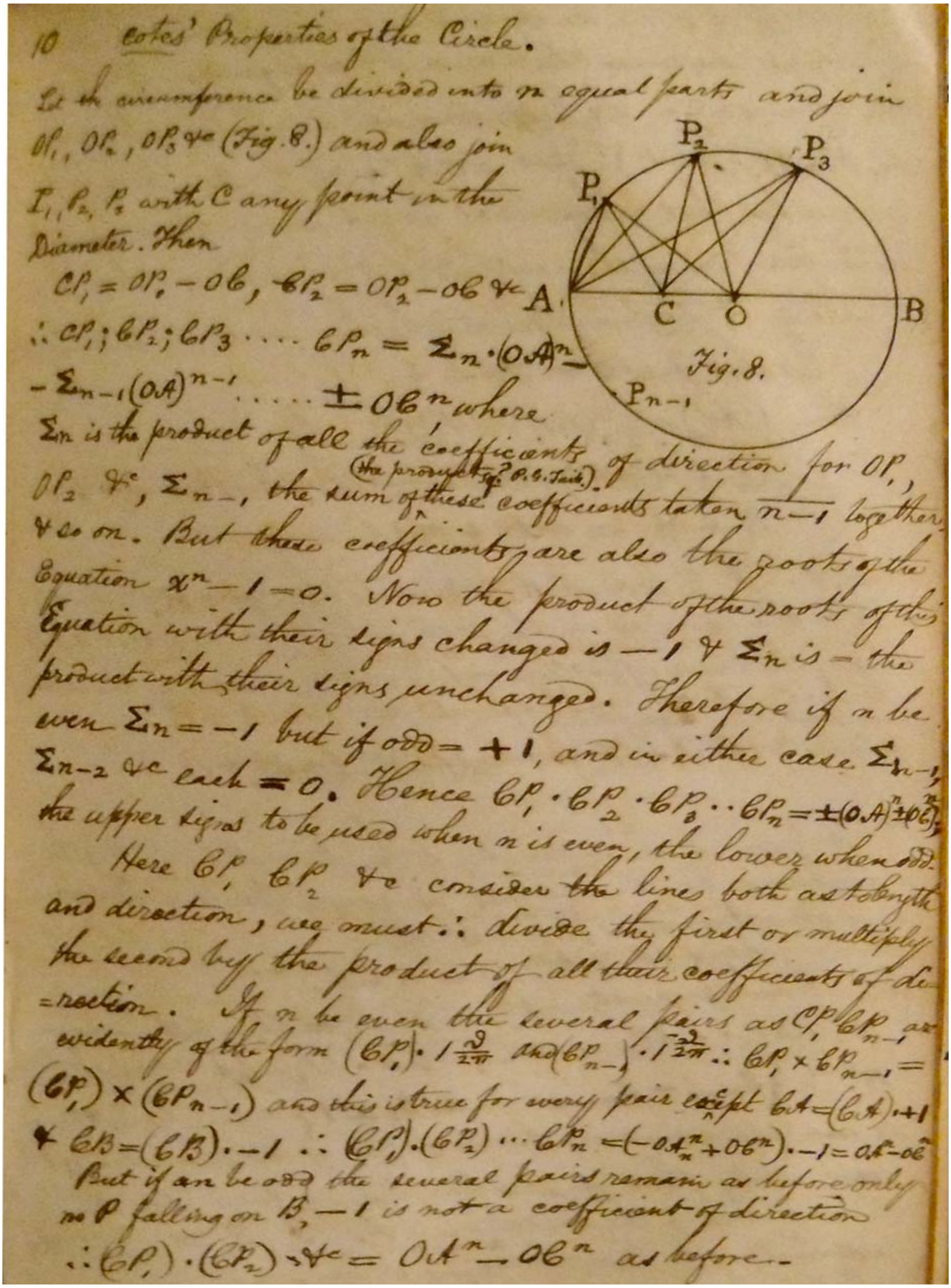
but $1^{\frac{A+B}{2\pi}} = \cos \overline{A+B} + \sqrt{-1} \sin \overline{A+B}.$

Equating then the sinal & cosinal parts of these, we have

$\cos A \cdot \cos B - \sin A \cdot \sin B = \cos \overline{A+B}$

$\sin A \cdot \cos B + \cos A \cdot \sin B = \sin \overline{A+B}$

Figure 5: An extract from Terrot's lecture; Tait's drawing of Figure 7 (Tait–Maxwell School-book)



10 Cotes' Properties of the Circle.

Let the circumference be divided into n equal parts and join OP_1, OP_2, OP_3 &c (Fig. 8.) and also join P_1, P_2, P_3 with C any point in the Diameter. Then

$CP_1 = OP_1 - OC$, $CP_2 = OP_2 - OC$ &c

$\therefore CP_1 ; CP_2 ; CP_3 \ldots CP_n = \Sigma_n \cdot (OA)^n - \Sigma_{n-1}(OA)^{n-1} \ldots \pm OC^n$ where Σ_n is the product of all the coefficients of direction (the product? P.G.Tait) for OP_1, OP_2 &c, Σ_{n-1} the sum of these coefficients taken $n-1$ together & so on. But these coefficients are also the roots of the Equation $x^n - 1 = 0$. Now the product of the roots of the Equation with their signs changed is -1 & Σ_n is $-$ the product with their signs unchanged. Therefore if n be even $\Sigma_n = -1$ but if odd $= +1$, and in either case Σ_{n-1}, Σ_{n-2} &c each $= 0$. Hence $CP_1 \cdot CP_2 \cdot CP_3 \cdots CP_n = \pm(OA)^n \pm OC^n$, the upper signs to be used when n is even, the lower when odd.

Here CP_1 CP_2 &c consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all their coefficients of direction. If n be even the several pairs as CP_1 CP_{n-1} are evidently of the form $(CP_1) \cdot 1^{\frac{\theta}{2\pi}}$ and $(CP_{n-1}) \cdot 1^{\frac{-\theta}{2\pi}}$ $\therefore CP_1 \times CP_{n-1} = (CP_1) \times (CP_{n-1})$ and this is true for every pair except $CA = (CA) \cdot +1$ & $CB = (CB) \cdot -1$ $\therefore (CP_1) \cdot (CP_2) \cdots CP_n = (-OA^n + OC^n) \cdot -1 = OA^n - OC^n$

But if n be odd the several pairs remain as before only no P falling on B, -1 is not a coefficient of direction $\therefore (CP_1) \cdot (CP_2)$ &c $= OA^n - OC^n$ as before.

Figure 6: An extract from Terrot's lecture; Tait's drawing of Figure 8 (Tait–Maxwell School-book)

B Editorial corrections

The following table records the necessary editorial corrections made to the transcription:

Reference	**Editorial correction**
§2, pg 1	Tait has '$\angle ACA_1 = A_1CA_2^2 = A_2CA_3$ &c then calling $CA = 1$, $CA_1 = a$, CA_2 a^2'. I cannot see why Tait has the superscript 2 in 'CA_2^2' so I have ommited it. I have also added in an equals sign between 'CA_2' and 'a^2'.
Ex.1, pg 2	Tait has '$\because AC + CB = a \times \left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a = a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.' I have ommited the '$= a$' as I believe it appears only since there is a break in the line.
§7, pg 3	Tait has '$\because \left(\cos\vartheta + \sqrt{-1}\cdot\sin\vartheta\right)^p = \cos p\vartheta + \sqrt{+1}\sin p\vartheta$' which is incorrect: there should of course be a -1 under the second square root sign, rather than $+1$. The ink on the original appears smudged here. Perhaps Tait attempted to correct his error.
§9, pg 5	Tait has 'which is $\therefore = r^2 = (CA^2) = (CA_1)^2$'. I have repositioned the superscipt 2 to sit in its proper place, outside the bracket '(CA)'.
§10, pg 5	Tait has 'and this is true for every pair except $CA = (CA)\cdot +1$ & $CB = (CB)\cdot -1 \therefore (CP_1)\cdot(CP_2)\cdots CP_n = (-OA_n^n + OC^n)\cdot -1 = OA^n - OC^n$' I have added in the bracket around CP_n which Tait has forgotten.
§11, pg 6	Tait has 'If $\overline{n-1}$ be even, $AP_1\cdot AP_2\cdot$ &c $= (AP_1)(AP_2)$ &c the several pairs of coefficients giving unity for their products.' I have added in $\cdot$ on the right hand side of the equation (as a sign of multiplication), as without it, Terrot's meaning is at first unclear.

Table 1: Editorial changes made to Tait's notes.